The Arithmetic Teacher

MARCH • 1959

Basic and Superficial Ideas in Arithmetic
MAURICE L. HARTUNG

Twentieth Century Mathematics for the Elementary School
H. VAN ENGEN

The Content and Organization of Arithmetic
G. T. BUSWELL

Arithmetic for the Gifted
CYNTHIA PARSONS

Achievement in England, California, and St. Paul
THOMAS L. BOGUT

THE ARITHMETIC TEACHER

a journal of
The National Council of Teachers of Mathematics

Editor: BEN A. SUELTZ, State University Teachers College, Cortland, N. Y.

Associate Editors: MARGUERITE BRYDEGAARD, San Diego State College, San Diego, Calif.; E. GLENADINE GIBB, State Teachers College, Cedar Falls, Iowa; JOHN R. CLARK, New Hope, Pa.; JOSEPH J. URBANCEK, Chicago Teachers College, Chicago 21, Ill.

All editorial correspondence, including books for review, should be addressed to the Editor. Advertising correspondence, subscriptions to THE ARITHMETIC TEACHER, and notice of change of address should be sent to:

The National Council of Teachers of Mathematics
1201 Sixteenth St., N.W., Washington 6, D. C.

Entered as second-class matter at the post office at Washington, D. C., June 21, 1954, under act of March 3, 1879. Additional entry granted at Menasha, Wis.

OFFICERS OF THE NATIONAL COUNCIL:

President: Harold P. Fawcett, Columbus, Ohio

Executive Secretary: M. H. Ahrendt, Washington, D.C.

Past President: Howard F. Fehr, New York, N.Y.

Vice Presidents: Robert E. Pingry, Urbana, Ill.; Alice M. Hach, Racine, Wis.; Ida May Bernhard, Austin, Texas; E. Glenadine Gibb, Cedar Falls, Iowa

DIRECTORS: Clifford Bell, Los Angeles, Calif.; Annie J. Williams, Durham, N.C.; Robert E. K. Rourke, Kent, Conn.; Phillip S. Jones, Ann Arbor, Mich.; Philip Peak, Bloomington, Ind.; H. Vernon Price, Iowa City, Iowa; Frank B. Allen, La Grange, Ill.; Burton W. Jones, Boulder, Colo.; Bruce E. Meserve, Montclair, N.J.

THE ARITHMETIC TEACHER is published six times per year, in October, November, December, February, March and April. The individual subscription price of $5.00 ($1.50 to students) includes membership in the Council (For an additional $3.00 ($1.00 to students) the subscriber may also receive THE MATHEMATICS TEACHER.) Institutional Subscription: $7.00 per year. Single copies: 85¢ each. Remittance should be made payable to the National Council of Teachers of Mathematics, 1201 Sixteenth St., N. W., Washington 6, D. C. Add 25¢ per year for mailing to Canada, 50¢ per year for mailing to other foreign countries.

Table of Contents

		Page
Distinguishing Between Basic and Superficial Ideas in Arithmetic Instruction	*Maurice L. Hartung*	65
Twentieth Century Mathematics for the Elementary School	*H. Van Engen*	71
The Content and Organization of Arithmetic	*G. T. Buswell*	77
Arithmetic for the Gifted	*Cynthia Parsons*	84
A Comparison of Achievement in Arithmetic in England, California, and St. Paul	*Thomas L. Bogut*	87
Why Is Arithmetic not the End?	*W. W. Sawyer*	95
Levels of Difficulty in Division	*Arden K. Ruddell*	97
Children's Success with Two Methods of Estimating the Quotient Figure	*Frances Flournoy*	100
The National Council's Committee on Arithmetic	*J. Fred Weaver*	104
Helping Children Understand Verbal Problems	*Milo K. Blecha*	106
Sources of Articles Published in THE ARITHMETIC TEACHER	*Agnes G. Gunderson*	108
What about Those 'Zero Facts'?	*Robert A. Oesterle*	109
Art and Arithmetic	*Elizabeth Ragland*	112

THE ARITHMETIC TEACHER

Volume VI Number 2

March 1959

Distinguishing Between Basic and Superficial Ideas in Arithmetic Instruction*

MAURICE L. HARTUNG

The University of Chicago

CRITICS OF OUR SCHOOLS complain that not enough arithmetic is learned by children in general, or, at least, that some children fail to learn enough. I believe these views are well-founded. Most children can and probably should learn much more arithmetic than they do.

In taking this position I am not aligning myself with irresponsible critics who assert that the schools are failing miserably. Competent and unbiased observers cannot avoid noticing that many children learn substantial amounts of arithmetic in the schools. My position is based upon the belief that the goals and methods of the schools should constantly be reexamined in the hope that the efficiency of the instructional process can be improved.

Instructional programs in arithmetic are encrusted with layer on layer of ideas and practices that have accumulated through the years. These ideas and practices can be examined critically from the point of view of modern mathematics. They can also be examined from the point of view of modern learning theory. When these analyses are carried through systematically, some of the ideas and practices of current instructional programs are seen to be questionable either on mathematical grounds, or on psychological grounds, or both. We see that the need today is not just to teach more topics in arithmetic, or to provide more drill, or to assign more homework. Instead, we do need more emphasis on fundamental concepts. We do need better organization of the arithmetic program for learning purposes. As teachers, we do need to distinguish between what is basic and what is merely superficial in an arithmetic program.

Means and Ends

Much of the inefficiency in our instructional program arises from a confusion between means and ends. I shall not enter into the philosophical problems that cluster around these concepts, such as whether evil means are justified by good ends. However, I do want to emphasize that when we focus much of our attention upon the means by which certain ends may be achieved, we do not necessarily improve our efficiency. For example, the computational abilities to be developed in arithmetic should be regarded as means, and not as ends. Computation for its own sake is fruitless, but computation is often essential as a means to the end of obtaining a solution of a problem.

* An address delivered to The National Council of Teachers of Mathematics, Elementary Section, New York, December 29, 1958.

Now in many schools almost all of the energy put into the arithmetic program is devoted to the computational aspects. The more basic objectives related to problem solving are relatively neglected. The result of an unbalanced distribution of energy is that achievement in arithmetic is less than it might be.

In order, then, to distinguish between what is basic and what is not, we must be aware of the objectives of the program. I shall make no attempt here to give a comprehensive formulation of objectives and to defend it. I merely assert that *the ability to solve problems is paramount.* To achieve this general objective most efficiently the pupil must acquire certain more specific behaviors. He must, for example, understand mathematical concepts, recall important items of mathematical information, recognize mathematical relations, and skillfully carry through mathematical processes. He must be able to apply these and many other more specific behaviors to problematic situations new to him, and do this in an integrated fashion which yields solutions to the problem. It is desirable that while the pupil is learning to achieve the main objective, he should also begin to acquire an appreciation of mathematics itself as an intellectual discipline. Throughout the remainder of this discussion these objectives will be regarded as basic criteria for evaluating certain ideas and practices of instruction in arithmetic.

Discovery versus Authority

Let us note, first of all, that now-a-days we all agree that arithmetic should be taught meaningfully. Nearly everyone accepts the proposition that pupils should understand the concepts of arithmetic. Hence, given two instructional methods, if one of them makes arithmetic more meaningful to the pupils, or produces greater understanding, we should prefer that one.

There have been a number of investigations which have compared the achievement of pupils who were taught by certain experimental procedures with the achievement of similar pupils taught by "traditional" methods. Almost invariably it turns out that pupils taught by the experimental method do better than, or at least as well as, the others when both are judged on the basis of standard or comparable test scores. This occurs in spite of the fact that the teachers as a rule were more experienced in using the traditional procedures. As a result of several such studies, we now favor methods which rely upon pupil "discovery" and generalization of basic facts and principles in preference to methods which present the facts and principles in more or less authoritarian fashion and seek by drill to establish them in the pupil's mind. The available evidence indicates that pupils taught by "discovery" methods have better understanding, better retention, and better appreciation of the arithmetic they have learned.

Before going on, let me make it clear that I heartily endorse meaningful teaching and condemn as outmoded the sterile method which relies upon rote or mechanistic learning. But I must call attention to the fact that when it comes to choosing the best among variant meaningful methods we have very little in the way of research findings directly *ad hoc* to fall back upon. Even if we know that Method X is better than Method A, and Method Y is better than Method A, and Method Z is better than Method A, this information alone does not enable us to arrange Methods X, Y, and Z in order of merit. Although a "discovery" method may be superior to a rote method of learning, how shall we choose among several proposed "discovery" methods?

In making such a choice we should, I think, examine the extent to which the method leads quickly to the discovery of basic concepts or facts and does not waste valuable instructional time by encouraging the discovery of numerous facts and relations which are of limited significance.

In the science of arithmetic there are a great many facts and relations that may be "discovered" and studied by the pupils if appropriate methods are used. Relatively few of these facts and relations are, however, of basic importance. For example, it is inter-

esting to observe that the sum of 2 and any odd number is the next greater odd number, but this is not really a basic generalization. It is superficial. If experience in discovery is the main goal of instruction in arithmetic, we can afford to spend considerable amounts of time in providing this experience. But I maintain that experience in discovery is not a basic objective of the arithmetic program. It is only a *means* to other important ends. We must, therefore, select carefully the places in the arithmetic program where "discovery" by the pupils is to be encouraged, and we must guide the pupils toward fundamental procedures in their learning activities.

Suppose, for example, the objective is that the pupils learn the basic facts of addition for which the sums exceed 10. Briefly, this means that the pupil must learn to replace 8 plus 7 by 15, 9 plus 3 by 12, and so on, until he has learned 34 other similar tasks. Faced with the task of learning $8+7=15$, the pupil in some programs is encouraged to think as follows. "I do not know what 8 plus 7 is, but I do know that 7 plus 7 equals 14. Now 8 is 1 more than 7, so 8 plus 7 must be 1 more than 14. Therefore $8+7=15$." This is clearly good reasoning, and it illustrates a method of "discovering" an unknown number fact. It is not, however, economical of time and thought, and is decidedly less general in application than other possible methods of dealing with the same situation.

A basic and general approach to teaching addition must emphasize the principles of the Hindu-Arabic numeral system with "ten" as the base. From this point of view the replacement of 8 plus 7 hinges upon a regrouping which yields 10 plus 5, and by the place-value principles of the numeral system this is written briefly in the form "15" and read "fifteen." The same principle may be used with each of the 36 facts, and later may be extended to the addition process in general.

Now the point at issue here is not a mere difference of opinion as to how these addition basic facts are to be learned. This is only an example. The central issue is whether the focus of attention is to be upon a limited set of basic concepts and principles which have wide application—in this case the principles of the numeral system—or whether it is to be upon methods which are relatively particular to the situation at hand. The assumption is that meaningful learning and "discovery" are to be encouraged in any case. But this example shows how undue concern for and emphasis on a method can distort the program to the point where superficial ideas or uneconomical procedures are being stressed for the sake of exploiting the method.

Concrete Aids

The point that has just been made may perhaps be driven home by considering what frequently happens when an effort is made to devise concrete aids to help pupils learn arithmetic. To keep it simple, suppose the basic purpose of the aid is to help pupils become acquainted with the individual numbers from one to ten, and the relations of order between them. The aid takes a concrete physical form. It must be made of some material. The objects used must have some shape, some size, some color. Often they have some fixed arrangement, as on the abacus. The device may be cheap, or it may be expensive in terms of the time required to make it or the money required to purchase it.

Now recall the comment made earlier that almost any meaningful method will secure better results than a non-meaningful method. We can expect the builder of the aid to find that it increases the effectiveness of teaching when used in situations where no comparable device has been in use before. In general, we will find that he is enthusiastic about it. Some of his pupils may be enthusiastic also. I am prepared to accept the conclusion that the device, if properly designed and used, is effective.

We should observe that the very qualities that make the device effective—the material, the color, the physical arrangement of the objects—are at the same time limiting. Color is useful for the purpose of identifying

particular objects in a collection, but it has no numerical significance. The pupil who sees or imagines a set of objects and seeks to answer the question "How many?" must cast aside every property of the particular set of objects except their numerousness. He must learn to ignore size, shape, weight, color, and all other physical aspects of the set of objects, and of all equivalent sets, to arrive at the abstract concept of number. To help him in this, we should use varied collections of materials, and not be limited by the particular qualities of any special learning aid.

Let us also note that once a device has been built for a particular purpose, there is a tendency to explore its uses for other purposes. For some of these purposes it may serve very well, but for other purposes it usually has severe limitations. Nevertheless, there is a temptation to force it into the situation, even if the mathematical aspects of the situation have to be distorted to fit the manipulative characteristics of the device. Thus it often happens that this sort of *means* tends to be looked upon as an end in itself, and attention is lavished upon it. What emerges is not a coherent arithmetic program, but a series of episodes. It is as though the author of a play has devised the plot to fit the scenery available, instead of designing the scenery to enhance the effectiveness of the plot.

Up to this point I have been saying that it is imperative that we identify the basic ideas that underlie the arithmetic program, and hew to the line in getting them into the minds of the pupils. We should not allow ourselves to become so absorbed in efforts to devise or use special instructional methods and devices which, after all, are only means to an end, that undue amounts of pupil time and energy are diverted from the main task. During the remainder of my time I want to talk about the most basic task of all—the teaching of problem solving.

Problem Solving

In teaching problem solving, just as elsewhere in our current programs, we have tended to rely upon superficial rather than fundamental ideas. As evidence of this I cite the excessive amounts of time, energy, and confidence that are expended on attention to "cue words" in trying to analyze verbally formulated problem situations. I have discussed the teaching of problem solving with many hundreds of teachers in small group meetings. Usually I state one or two typical problems as examples, and ask how they are to be solved. Then I ask the teachers to tell me how they decided what they should do to solve the problems. Almost invariably the responses show a preoccupation with the "cue word" approach. Before I can get attention focused on basic questions, I must convince these teachers that the "cue word" approach is superficial, is often confusing, and may lead pupils astray. Under the circumstances it is not at all surprising that the pupils also rely upon "cue words."

A basic approach to problem solving in arithmetic must help pupils recognize the actions, real or imagined, that occur in the problem situation. It should then teach them how to express the situation in symbolic form. In most cases that are suitable for elementary school this symbolic expression takes the form of a simple equation. Then the pupil must learn to solve the equation, and it is through this basic means that computational processes appropriate for solving the problem are recognized. This is in contrast to teaching which requires the pupil to recognize the computational processes without a basic method or the aid of symbolism, and to plunge immediately into computation.

For example, consider the following problem.

> Bill had 43 pennies in his pocket. He spent 19 pennies. How many pennies did he have left?

Everyone readily agrees that this is "a subtraction problem." If asked why he thinks so, a person usually says: "Because he spent some money. We want to find how many are left." If asked to describe a dramatization of this situation, teachers say: "Put 43 pennies on a table. Take away the 19 pennies

he spent. Count how many are left." With all of this I am in full agreement.

Suppose now we consider another problem stated as follows.

> Bill had some pennies in his pocket. He spent 19 pennies. He had 24 pennies left. How many pennies did he have at first?

This problem sounds rather similar to the other one. However, most persons say this is an addition problem. If asked to explain why this is an addition situation, they often falter. If I point out that the sentence: "He spent 19 pennies" is identical in both problems, and that in the first problem they argued that this suggests a subtractive situation, but now they argue that addition is involved, they become confused.

Finally, if asked to describe a dramatization they say: "Put a group of 19 pennies on the table, and another group of 24. Put the 24 group with the 19, and count the total group. The important thing to notice here is that the proposed dramatization does not follow the original script at all. Instead, it follows a new script designed to fit the action to the mathematical solution of the problem rather than to the original situation that was described.

If now we express the first situation in symbolic form we obtain something like the following:

$$43-19=? \text{ or } 43-19=n.$$

The situation was, basically, subtractive, and the symbolism reflects this clearly. If we express the second problem situation in symbolic form we *should* obtain the following:

$$?-19=24, \text{ or } n-19=24$$

This formulation recognizes that the original situation is again basically subtractive. To give a true dramatization the teacher should put 43 objects in a box or sack so that the number cannot be immediately determined. Then 19 should be withdrawn or taken away to represent the 19 pennies that were spent. Then the class may be allowed to count the remainder and find that there are 24. This dramatization of the situation is, of course, quite different from the one proposed a few moments ago.

At this point we should note that we have arrived at a true symbolization of the situation, but still do not know the answer. We cannot actually subtract, since the minuend is not known. We must find an indirect method of solving the problem. As teachers, we may now invoke a general procedure. The suggestion is that we try changing the situation back to what it was. This means that, to the 24 that remain, we put back the 19 that were taken away. If this action is dramatized and symbolized, we obtain the new equation $24+19=n$, and this is readily solved by addition. Thus we see that, although the original situation was fundamentally subtractive in its structure, the solution of the problem is achieved in this case by the addition process.

This discussion illustrates how superficial methods, such as reliance upon "cue words," can mislead. It shows how a basic method—namely, the use of an equation—permits us to symbolize the problem situation in the way it is most naturally viewed, and leads us to the solution. The discussion also illustrates how, unless we are careful, we can let a preconceived idea about the solution lead us into a dramatization that fails to clarify because it does not correspond to the actual structure of the original problem situation.

It should be unnecessary to say that the sort of basic thinking I have here described should permeate the teaching of problem solving. From simple beginnings like the examples cited above, it should grow in strength and in scope until pupils can solve increasingly complex problems. We have been asking children to solve problems without providing them with the symbolic tools and analytical methods that have been invented to make this task easier. It is doubtful if we will ever achieve the fundamental objective of mathematical instruction— that is, the ability to solve problems— until we recognize that the use of equations and similar basic symbolic aids is at least as important as demonstrations with concrete materials of limited utility.

In this connection I must briefly call attention to the fact that a very substantial portion of all problems in mathematics involves rates and comparisons which lead to proportionality relations expressed in terms of ratios. Fractions also involve such relations. These are really basic ideas, but at present they are not explicitly taught in most schools. Instead, problems involving these ideas are solved by various special rules or short-cut methods, and valiant efforts are made by conscientious teachers to make these methods meaningful. They succeed only in part, or superficially, because the analysis does not reach the heart of the problem. Here again it is important to distinguish what is basic from what is superficial, and to start explicitly teaching the basic concepts.

In conclusion, let me say that I see great possibilities for improvement in arithmetic instruction if we are willing to acknowledge that some of our teaching has been superficial at best. As I said at the outset, the remedy is not more drill, but deeper insight as to what is really basic in arithmetic.

EDITOR'S NOTE. Professor Hartung argues that discovery is not an aim in itself but that we should use good methods of discovery in terms of the mathematical situations with which we are dealing. It may be argued that these very methods of discovery however form the skeins from which other discoveries are made and that the spirit of discovery, if not the method, may well be considered an aim. It is this spirit that vitalizes learning and a method of discovery may well be the more valuable residual of the learning. Professor Hartung's point about using visual and manipulative materials for situations where they really enhance learning is well taken as is the caution that learning should not be dependent upon a specific device or color or form. He cautions against the use of cue words in seeking a solution to a problem and suggests that we seek the essence of the relationships in the problem and express this is simple equation form. That is, we should think before rushing into computational work with paper and pencil. Our whole shift in the teaching of arithmetic during the past twenty-five years has been from a mechanical method fostered by drill to one of thinking leading to discovery and understanding. INSIGHT is a good key word.

DISCOVERING MEANINGS IN ARITHMETIC

by Grossnickle and Brueckner

Designed to increase classroom effectiveness by emphasizing the teaching of Arithmetic meaningfully.

The John C. Winston Company
1010 Arch Street
Philadelphia 7, Pa.

CONTENTS

Introduction: The Modern Arithmetic Program
Our Decimal Number System
Organizing the Arithmetic Program
The Arithmetic Classroom as a Learning Laboratory
First Steps in the Teaching of Arithmetic
Teaching Basic Facts in Grade Two
Addition and Subtraction of Whole Numbers
Multiplication of Whole Numbers
Division of Whole Numbers
Addition and Subtracton of Fractions
Multiplication and Division of Common Fractions
Decimal Fractions
Quantitative Thinking and Problem Solving
How to Teach Measurement
Evaluation in Arithmetic
Diagnosis and Guidance in Arithmetic
The Enrichment of Learning in Arithmetic
Appendix—How to Prepare Essential Materials and Learning Aids
Index

Box 265, Marlboro, Mass.
Blanche Building, Marianna, Fla.
703 Browder Street, Dallas 1, Texas
190 Waverly Dr., Pasadena, Calif.
5641 Northwest Highway, Chicago 46, Ill.

Twentieth Century Mathematics for the Elementary School*

H. Van Engen
University of Wisconsin, Madison

Recent events have brought the problems of mathematics instruction to the attention of the American public. In so far as the public is concerned, this has been a post-Sputnik development; however, many were aware that our mathematics program was in need of repair prior to Sputnik I. One need only recall the existence of committees and commissions appointed prior to Sputnik I to document pre-Sputnik awareness of the need to examine carefully the mathematics programs in our schools and colleges.

In order to place the elementary school problems in proper perspective, it would be well to remind you that those who approach our problems rationally do not consider that all the problems of mathematical education are located in the elementary school.

Instructional problems pervade the educational structure of the schools and colleges of America. The very existence of committees, appointed by academically respectable organizations, who are directed to study one or more segments of our instructional program, and all segments of the program are within the domain of one or more of these committees, supports such a statement. The existence of these committees refutes the charge of some individuals that a certain subset of our total teaching personnel is responsible for the lack of mathematical literacy. Such positions regarding the status of our educational problems can only be taken by individuals who are misinformed as to the nature of the overall problems in mathematical education.

In part, our present difficulties with the elementary and secondary mathematics program is due to the failure of those who are knowledgeable about mathematics to be interested in the total program for educating teachers. This lack of interest has allowed a program to develop which some people describe as antiquated and even the strongest supporters of the present program will concede that it needs attention. We are now reaping the harvest resulting from the mathematical world's complete lack of interest in the program of the elementary schools. Mathematicians have suddenly discovered that during recent times when the mathematicians' concept of mathematics was in the process of change the mathematics of the elementary school changed in a fashion which is inimical to the best interests of mathematics and, consequently, inimical to the best interests of children. For just a brief moment let us look at this situation.

The Changed Concept of Mathematics

In the last quarter of a century, mathematicians have come to conceive of mathematics as the study of structures. This is particularly true of algebra. Structure as a mathematician uses this word is a difficult concept to define, but it can be described as as the search for patterns; patterns which can be used to arrive at solutions to problems which pervade the social sciences, physical sciences, biological sciences, our military problems, in fact, problems in every area. If a mathematician is called in on a problem, he will ask himself, "Does this problem fit any of the patterns I know about?" If it does, he immediately sets about to apply the pattern to arrive at a solution. If the problem does not fit any previously known pattern, it becomes a challenge to devise such a pattern. The advantages of this method can be perceived even by the most mathematically

* An address given at the Christmas meeting of the National Council of Teachers of Mathematics, December 29, 1958.

unsophisticated individual. This approach solves whole classes or groups of problems simultaneously. It is a powerful method which rests on abstractions and generalizations of a high order. It is this ability to see patterns in similar and seemingly diverse situations that the schools have failed to nurture.

Time does not permit a fuller development of this idea. However, more will be said about structure and generalizations in the discussion of problem solving which is to follow.

Fragmentized Arithmetic

What happened to arithmetic while mathematics was being developed as the study of patterns? In the twenties and thirties, the period during which modern algebra was blossoming, arithmetic was being psychologized into myriads of little items. It was fragmentized to such extent that one research study of this period points out that to teach the addition of proper fractions some eighty different computational techniques should be taught. The instructional theory based on this concept of learning was easy to comprehend although difficult to put in practice. To teach addition of fractions it was sufficient to drill the child on each of the eighty techniques. As a basis for teaching mathematics, this concept of the task of the teacher is as revolting as it is simple.

We could look upon this as a rather amusing, even though pathetic, bit of history in the teaching of arithmetic if it were not for the fact that the arithmetic programs in our schools still make considerable use of this over-simplified concept of the learning process. One need only cite the forty-odd times we teach "just a little bit new" about the process of division. Many other examples could be given but space will not permit.

Social-Utility Arithmetic

The last quarter century also produced another movement in arithmetic which is definitely unsound mathematically and psychologically. The "social-utility" movement added to the confusion created by the fragmentization of arithmetic. It was, and is, widely accepted by many elementary teachers. The social-utility aspect of arithmetic instruction has many proponents each advocating somewhat similar aspects of the position. We need not review the various tenets of this position. It is sufficient to say that for the schools of today this philosophical position is definitely unacceptable. It is unacceptable because we do not have a crystal ball which tells us what is useful and because children like mathematics because it is mathematics and not because in later years it may turn out to be a very useful tool for them. The real crux of the situation rests far from the point at which the social-utility people have their eyes focused. Up to that point to which a child can understand the basic principles of mathematics, he will have no difficulty making applications of mathematics to daily life. Children encounter trouble with the applications when they do not understand mathematics, or when they are taught a "nonmathematical-arithmetic."

Articulation with Mathematics

These movements have resulted in an arithmetic program which has not led smoothly into the study of the advanced reaches of mathematics. In spirit the present program is a complete stranger to mathematics; in content, it lacks modernity; in its exclusive attention to computational aspects of the subject, arithmetic as practiced in the schools, is not mathematics. Let us examine more closely some of the needed changes in the content of the arithmetic program in our schools.

Numbers Have Many Names

Arithmetic gets a poor start in the first grade when teachers fail to recognize that numbers have many different names and that it is important to study the patterns of these names and the sense conveyed by a name. An illustration may help clarify this idea.

If two apples are on a plate and we put three more apples on the plate, the number

of apples on the plate may be symbolized by '2+3'. '2+3' carries the sense of the event described above and it is a name of the number which is the answer; However, it is not a standard name for the answer. The standard name is '5'. On the other hand, the symbol '3+2' is another name for the same number, but its sense is different from that of '2+3.'

On the surface there may seem to be little new in this, but let me point to the following.

1. The child should recognize that 3+2 is a correct answer to the problem described by the statement "put two apples with the three apples on the plate," and that in this instance he should not use the symbol '2+3' because the sense of this symbol more closely fits the problem described by the statement "put three apples with two apples." Note that we teach the child to observe the structure of the physical situation and to choose an appropriate symbol describing the events. He then selects a standard name for the symbol.

2. Our instructional procedures are such that we distinguish between the name of a number and the number itself. Of course, the child is not burdened with this distinction even though he should learn that numbers have many names.

3. We are on the threshold of an important mathematical idea, namely, commutativity.

4. This illustration puts the so-called basic facts and all arithmetic computation into proper perspective. Computation in arithmetic is merely a collection of name changing devices. We accept as answers such symbols as $14+49$, 32×156, $\frac{1}{2}\times8$, and $8\div2$ provided the sense of the symbols are maintained, but we need a way to find a standard name for such answers. Parenthetically, let me remark that this point of view immediately necessitates discarding some very common, but fallacious, ideas in arithmetic; ideas exemplified by the dictum, taught in high circles that $\frac{1}{2}\times8$ is the same as $8\div2$. If one means by this that $8\div2$ and $\frac{1}{2}\times8$ can be used as names for the same number, then there is little to quarrel about except that the observation is not very profound. After all, there is an endless array of names for this number. Here are just a few: $2+2$, $5-1$, 2×2, $16-15+3$, ad infinitum. On the other hand, if one means that the sense of the symbols $\frac{1}{2}\times8$ and $8\div2$ are the same then the statement is utterly erroneous and as such can only confuse the child.

Number Pairs

Schools have not exploited the use of number pairs as they occur in common everyday experiences and in many instances where they have been exploited, the various uses have not been clearly distinguished so as to enable the child to grasp their significance. An illustration of what I have in mind will help.

Suppose we have a pile of apples which are to be divided so that for every two apples I get you get three apples. This is a simple use of a pair of numbers, often called a ratio. Note that even though it is written like a common fraction, it is not a common fraction. This use of ratio is not difficult for even the kindergarten-primary grades and it leads to some very important mathematical ideas. In the illustration, the child soon sees that the pile of apples could have been divided between the two of us more expeditiously by my taking four apples for every six apples he takes or six for every nine and so on. In other words, he learns that there are many, many number pairs that express the essential idea of the problem. Here again we are dealing with the many names for a mathematical entity. In this instance, instructional procedures should be so orientated that the number pairs are collected into a set. Furthermore, the child learns a test for determining the family to which a given number pair belongs.

In this problem, we are dealing with what a mathematician calls an equivalence class. However, the names "equivalence class" and "set" or "family" are not important. It is important, however, that early in the elementary school the child learns to use pairs of numbers as ratios in an intelligent way

and in such a way that the road is open to more powerful uses of the fundamental mathematical ideas in later years.

Time prevents an extended discussion of number pairs as ratios. It may be profitable, however, to point out that every multiplication and division problem involves three numbers. Two of these numbers express a ratio and the third number is one term of a second ratio. For example, consider the problem: If John earns \$3.00 in one day, how much will he earn in 5 days. The phrase "\$3.00 in one day" means \$3.00 per 1 day and is written 3/1. Now we want to find another name for this ratio whose second term is 5. In other words, $3/1 = n/5$.

The "many names for the same number" approach is particularly fruitful in overcoming some of the difficulties encountered in the fundamental processes as applied to common fractions. Furthermore, it leads naturally to the consideration of the number line in fifth and sixth grade work—another idea that is much neglected but space prevents any discussion. We must pass on to the next item.

Numeration Systems and Number Systems

Closely associated with some of the ideas presented so far is the distinction between a numeration system and a number system. In the world of the elementary teacher, the term "number system" is used in the "numeration system" sense. The ideas of a number system are thoroughly neglected, in fact, even numeration systems are quite thoroughly neglected in elementary instruction.

In mathematics, we are faced with the problem of providing standard names for many numbers. Unless some systematic way is devised, the task of remembering the names would be beyond human ability. This system for naming numbers is a numeration system. One must remember, and children should be taught, that there are many numeration systems. Some teachers do yeoman work with numeration systems. On the other hand, when the mathematician thinks of number systems he thinks of such properties as associativity, commutativity, and the distributive laws. It is the latter that are inadequately treated in the elementary and secondary schools. True, the elementary schools do call the child's attention to such facts as $3\times2=2\times3$, but little is done with these ideas beyond learning the facts of arithmetic. Such a fundamental idea as the distributive law is completely ignored. There are, of course, reasons for this state of affairs, but are they sufficient reasons?

Problem Solving

The schools have not been successful in devising a sensible approach to problem-solving. This audience is all too familiar with the various proposals for improving the problem-solving ability of the elementary school pupil. In spite of all the proposals and the research, it is probably not too far amiss to summarize the results of present-day research by the single statement: The best way to teach children how to solve problems is to give them lots of problems to solve. Certainly a fresh approach to problem solving is needed.

There is some reason to believe that the failure to achieve any degree of success on the part of the various proposals for improving problem-solving ability is that the attention of the child is directed to the wrong element in the problem, namely, its answer. Now this may seem paradoxial, but it is put forth as a serious criticism of our efforts to teach children to solve problems. Arithmetic has been so "answer-minded" that teachers have forgotten that "first things must come first." This "answer-mindedness" is in keeping with the mistaken idea of what arithmetic is all about, namely, computation. Some twenty-five years ago, the idea was rather widely held that one should not attempt problem solving until all the basic facts had been memorized. After all, how could one solve a problem if he could not compute. While this attitude is not held today, we are still not far from it.

In our previous discussion, we have hinted at a more sensible and mathematically sound approach to problem solving.

You will recall that we said that '3+2' and '2+3' were different names for the same number, but that these symbols have different senses. Let us follow this thought a little further.

Consider these two problems.

1. Mary has six apples. Her mother gave her five more apples. How many apples does Mary now have?
2. Mary has some apples. Her mother gave her five more apples. Mary now has 11 apples. How many apples did Mary have?

There can be no doubt that the symbol '6+5' is an appropriate symbol to transmit the sense of the situation in the first problem. But it is customary to change its name, and we indicate this by writing $6+5=N$, where the N says to the child, "Find another name for 6+5 and use it to replace N."

However, in the second case the situation is entirely different. A little thought will convince you that $11-5$ is not a good symbol for communicating the sense of this problem. The sense of this problem is not that of removing five apples. As we look at the two problems we begin to feel that whatever symbols are used for these problems they should possess some similarity, because the two problems are similarly structured as to physical events. After examining the problem, one feels that the symbol 'N+5' preserves the sense of the second problem and that this symbol must be another symbol for 11. So we write

$$N+5=11$$

This approach teaches children to look for the sense of a problem and to select symbols which express this sense. In other words, we want the child to grasp the structure of the problem before he looks for the answer. The answer is obtained by finding the proper replacement for the place-holder symbol in the equation.

Certainly the basic differences between good problem-solvers and poor problem-solvers must reside in differences in ability to recognize the element which we have called structure. The good problem-solver knows "what's going on" in the problem even though he may not have been taught appropriate means for expressing this "going on." However, a knowledge of events leads the good problem-solver to the right answer. An "answer-minded" orientation in arithmetic has prevented us from giving the child a means for expressing these "goings-on" in problems and, consequently, the child has no systematic way to arrive as to how to get the answer. Experience has shown that no reliance on words and cues will ever help the child to make these decisions.

The method of problem solving we have illustrated here is a mathematician's approach to problems in miniature. One first searches for the fundamental structure of a problem situation; then he finds the appropriate symbols to express this structure. Once the problem has been structured, a knowledge of previous problems and problem-solving techniques can be applied. Certainly, no "cue" method or mere admonitions to THINK hold the mathematical power that the search for the structure of the physical situation can command. The failure of the older methods over the past years should be reason enough to banish them from the classroom and search for methods with more mathematical power.

Arithmetic as Patterns Vs. Drill

I have tried to convey to you that the study of mathematics is a constant search for patterns and that this search involves the ability to abstract and to generalize. This fact should cause all teachers to look carefully at their classroom activities. In fact, with the change in content of the curriculum there must come an entirely different concept of instruction. No drillmaster will ever be a mathematics teacher. Drill can do little more than teach children how to change the names of numbers. While this must be attained, it is not the end but only a means to an end. Once teachers sense this, there will be a profound improvement in both content and method of instruction.

I cannot leave this subject without commenting briefly on a few items closely connected with the subject under consideration.

Lock-Step Curriculum

Our arithmetic is predominantly placed to the abilities of the average or below average child. This condition has come about because of the pressure to teach arithmetic to the children of all the people. This is a very laudable objective, and most certainly we must teach arithmetic to the children of all the people. However, we cannot forget that we are obligated also to teach mathematics to the child of ability. There is no reason why the rapid learner should get caught in our lock-step curriculum except for inertia and a philosophy of education which declares that not to teach all children the same mathematics would be undemocratic. Some elementary schools are now experimenting with the rapid advancement of the academically talented child. This is a hopeful sign. There is no reason why talented children should not complete most of what is now considered to be seventh or eighth grade work in the first six years of school. Many well established patterns of thought about elementary education will have to be overcome before this can be done on a systematic basis, but it must come. Logic and common sense can provide no other answer.

If this day comes, it will mean that teachers of elementary arithmetic will need to know more mathematics. Teachers of today may know computation, but they don't know mathematics. Certainly, the improvement of the course content of arithmetic is in great part dependent on the increase in knowledge of mathematics on the part of the teacher.

Summary

We have tried to indicate to you how arithmetic can be revitalized by initiating a program which introduces some basic mathematical ideas into the elementary curriculum. However, new ideas alone will not revitalize the program. We must discard erroneous ideas about pupil promotions, grade placement of topics, and even erroneous mathematical ideas. Furthermore, basic philosophical and psychological positions must be modified. We must have good mathematics in the elementary school to enable the child to apply this mathematics in daily life; the applications alone will not teach mathematics. We must learn how to cope with wide differences of ability and we must learn how to teach lots of mathematics to the child with ability. And above all, we must get teachers to discard the idea that arithmetic can be taught by drill and drill alone. Abstractions, generalizations, a sense of structure cannot be "drilled" into a child. For this we need a fundamental change in our concept of what constitutes mathematics teaching.

It took years of indoctrination to establish present educational positions on arithmetic; it will take years of constant effort to reorient the program and the teachers to a position which carries mathematical power.

EDITOR'S NOTE. It is true that certain groups and individuals have constantly been calling attention to weaknesses in our elementary school program of arithmetic. The yearbooks, numbers Ten and Sixteen, of the National Council are exhibits. It is now twenty-five years since New York pioneered with "Mathematics for Elementary Schools" replacing "Arithmetic" in a move to discard the narrow view of the subject and the drill-memory approach to learning. But as Professor Van Engen points out, progress with the large majority of teachers is extremely slow and this in large part is due to the relatively low level of mathematical training of many of these teachers. Mr. Van Engen suggests that many pupils might well achieve our current grade seven and eight arithmetic by the end of grade six. We need again to ask, "What Is Grade Five (or any grade)?" If we agree that is means no more than the year following grade four, our view of the purpose of education might well be challenged.

The editor was impressed with Mr. Van Engen's introduction of the idea of "structure" and how this can revolutionize our approach to problem solving. This is a study of the essential pattern or relationship expressed in the problem. For example, the child who has generalized a pattern sufficiently so that he senses that "2½ M at $11.75 per M" is essentially the same as "How much will 3 pencils cost when the price is 8 cents each?" has gained an awareness of and an insight into situation-structures. Implicit in Mr. Van Engen's view of structure one finds "equation thinking" and "equation notation" as steps in procedure. But these need not be the formal equations one finds in grades eight and nine. In fact his pattern "2+3" is a step in "equation thinking."

The Content and Organization of Arithmetic*

G. T. Buswell
University of California, Berkeley

The content of this paper will be organized into three parts: first, two matters which are basic to any improvements in the arithmetic program, namely, the relation of content to individual differences and the effects of the deferred or stretched-out curriculum; second, some proposed changes at the primary and at the seventh-eighth grade levels; and third, some comments on materials of instruction and preparation of teachers. The issues to be covered are controversial and the reader will undoubtedly supply pros and cons to supplement the discussion given here.

Adjusting to Individual Differences

Teachers of arithmetic are aware of the stepped-up volume of criticism of the schools. This criticism is not spread equally over the entire school program but rather is focused sharply on the alleged failure of the schools to provide an adequate education for that segment of the school population that is going on to college and that will necessarily supply the intellectual and technical competence that our society requires. Certainly, the problem of individual differences is not new; the volume of published data on the subject during the last fifty years is impressive. Yet there is something different about the current criticism at least as it relates to the subject of arithmetic. It is not longer true that the arithmetic of everyday life is sufficient. The world of mathematics has moved swiftly in the last two decades. The work of a bookkeeper of a generation ago has little in common with his counterpart at present. The use of business machines and mathematical computers requires a different level of arithmetical understanding than was necessary when all computation was done by hand. The result is that the everyday arithmetic has been made simpler, as for example, when the modern market sums the total of purchases on a cash register tape which saves both the customer and the clerk the trouble of adding. On the other hand, there has developed a set of needs for a small portion of the population that are decidedly not "everyday" in character but are highly specialized. The present program in arithmetic has been reasonably successful in meeting the everyday needs of average pupils. But for the approximate third of the pupils who are headed for higher education and for work in technology and the sciences, both physical and social, it is quite a different matter. For these pupils arithmetic needs drastic revisions and additions. To be sure, the schools have done something about this, but it has been a feeble effort. It has been blocked on the one hand by the objection that in a democracy all pupils should be treated alike and on the other by the lack of administrative effort to make provisions for a real differentiation according to needs and abilities.

In a comparative study of arithmetic achievement of eleven-year-olds in England and California, published in the February 1958 Arithmetic Teacher, it was found that the 1077 pupils who constituted the upper third of the English sample made scores of 38 or above on the 70-item test. Only 13 California pupils from a sample of equal size made scores this high. It is this failure to produce superior achievement that is the chief cause for concern. That this difference is not limited to England and California is indicated in a recent letter from Dr. Herbert Spitzer saying that a doc-

* A paper read at the arithmetic section of the meeting of the National Council for Teachers of Mathematics in Cleveland, April 10, 1958.

toral candidate at the State University of Iowa had made a comparison of arithmetic achievement in Iowa and Holland. He found that the Iowa eighth grade was just about equivalent in arithmetic achievement to the Dutch sixth grade. He also reported that in Holland pupils have put in as much time on arithmetic by the end of the sixth grade as Iowa pupils have by the end of the eighth grade.

In view of the current pleas for strengthening the mathematics program, it is clear that the attempts to up-grade the arithmetic program for those pupils who expect to go to college must be pushed more vigorously. This will mean that they be put into special classes or sections where an intensive program in arithmetic can be carried on such as would be neither desirable nor possible with average and below average pupils. Obviously, there would need to be flexibility in the selection of pupils and in correcting errors in placement. These are matters of detail and are certainly no more difficult than selecting members of a first team for basketball or football. Until the principle of grouping pupils for learning in terms of interests and abilities in respect to what is to be learned is accepted, it is practically impossible to experiment with desirable modifications in the arithmetic program. But if rate of progress and content to be learned could be fitted to the abilities of the upper third of pupils, it might also be found that for such pupils the program in arithmetic could be completed in six years as is the case in many countries of the world. This would then make possible a program in high school mathematics that would really prepare students for the serious study that college students are presumed to undertake.

The Stretched-out Program

Another matter which affects the present situation is the internal organization and spacing of the arithmetic program. During the 1930's there was a good deal of discussion of what was called "deferred arithmetic." The controversy was set off by a series of three articles in the *N.E.A. Journal* in which it was proposed that the whole subject of arithmetic be postponed until the seventh year of school. Although few schools actually postponed teaching the subject to that degree, the discussion did result in a spreading or stretching-out of subject matter over longer periods of time. One result was that arithmetic was spread down into grades one and two where, up to that time, little systematic effort had been made to introduce the subject. This earlier beginning has now become quite general. There are probably some substantial advantages for average and below average pupils in this stretched-out program, but for more rapid learners it has only aggravated an already bad situation. For them it has resulted in padding topics already understood simply to fill time while the slower pupils catch up. Many of the topics and processes covered in arithmetic are introduced in other countries from one to two years earlier than is the practice here. It is especially difficult to justify much of the work in the seventh and eighth grades for the upper third of the pupils. It would be much to their advantage to cover all the topics of arithmetic by the end of the sixth grade and then go directly into a program of secondary mathematics appropriate for bright pupils of that age. The concept of a stretched out program needs to be re-studied. It was devised for average and slow learners. For the upper third of the pupils it constitutes a serious obstacle to interest and effort, and produces both boredom and dislike. For bright pupils, the routine topics of arithmetic can be covered more quickly than is now the case and the time saved can be used to extend the program to new kinds of content that are interesting and challenging. For the upper third of the pupils, the arithmetic program can be both enriched and completed earlier than is now done.

Changes in the Primary Grades

If the current criticisms of education result in a change in public opinion that will

permit a genuine effort to develop a program for bright pupils, and if schools can be persuaded to adopt, for this group, a systematically organized program covering six years, the needed changes would affect particularly the beginning grades and the seventh and eighth grades. Attention will be given first to the program for the kindergarten and grades one and two, especially for the upper third of the children in respect to intellectual and academic maturity.

The last twenty-five years has witnessed great gains in the primary arithmetic program. If one were to compare the present arithmetic material for first-grade pupils with what was available in 1930, the gains would be strikingly clear. Certainly the substitution of a concrete approach to number facts as contrasted with the earlier abstract drills will be approved by all. However, even now the systematic and coherent treatment of arithmetic usually begins at grade three and textbook publishers still designate as book one, the book for the third grade. What goes on in the kindergarten and in grades one and two is somehow thought of as outside of or preliminary to a really serious effort to deal with numbers. This should not be the case. What is needed is not a *formal* program of arithmetic at this level, but rather some serious thinking about a coherent total program beginning with the kindergarten. Formality and coherence are not synonymous terms. Specifically, it is now time to gather, and give careful study to, the many miscellaneous ways that have been proposed for introducing ideas of number. While the children who learn would of course not be aware of or concerned about the coherent development of arithmetic, the teachers certainly should be aware of it and should understand what is expected at each grade level. As a psychologist, I am aware of the position that the kindergarten should be preserved for social development. But experiences with numbers are a part of the child's world, and I know of no evidence whatever that an appropriate amount of effort to develop the child's interest in numbers will in any way harm his social development. The facts simply give no support to the position that learning basic concepts of numbers are out of place in the kindergarten. In many schools such practices have been carried on for years, greatly to the advantage of the children. If anyone is skeptical, he could not do better than to refer to the chapter by Olga Adams in the monograph *Arithmetic 1947*,[1] where she describes in detail the many practices followed in her kindergarten at the University of Chicago Laboratory Schools dealing with counting, fundamental processes, measuring, using fractions, telling time, and working with circles, squares, and triangles. There was nothing formal about Miss Adams practices, but they were certainly purposeful, seriously planned, and coherently developed. As a result, her pupils went on to the first grade with a rich background of number experiences that gave meaning and interest to their arithmetic.

Experiences with arithmetic in the beginning grades can be roughly classified into three groups. The first, and oldest, of these presented arithmetic as a set of facts to be learned with little appeal to understanding or to social uses. Abstract drills on number facts and number tables gave rise to the designation "formal" arithmetic. While some schools still follow these practices, it is hard to find anyone to defend them. A second group of practices uses concrete experiences as the proper way to introduce numbers to children. This is the most widely used method at the present time in the United States. It relies strongly on social arithmetic as a carrier of mathematical meanings. However, its aim is to lead pupils ultimately to these mathematical meanings. Much use is made of pictures, and usually in expendable work-book materials. A third way of presenting number experiences exemplified by the systems of Catherine Stern in this country and of Cruisenaire in Europe, pays less attention to socially concrete experiences and deals more directly with numbers and number relations. Al-

[1] University of Chicago Press.

though more evidence from controlled experiments on these methods is needed, they undoubtedly deserve careful study. Their use of manipulative materials is to be noted, as contrasted to the work-book type of exercises.

The weakness of the kindergarten-primary treatment of arithmetic is that it is too often looked upon as a readiness program rather than as an integral and important section of the total program of the elementary school. Primary arithmetic should be coherently related to the work that follows in grades three to six in such a way that number relations would be better understood and that many processes and topics could be introduced earlier. Much of the later aversion to the subject might be dispelled by a more carefully thought-out beginning.

Changes in the Upper Grades

The treatment of arithmetic in the seventh and eighth grades is a product of our eight-year elementary school and it affects the brighter pupils in a particularly adverse way. In this country it is customary for pupils who are headed toward higher education to cover the same program in arithmetic and to follow the same time schedule as do average and below average pupils. Whereas in European schools, students who plan to go to college begin their secondary school mathematics at age 12, we defer it until age 14. This results in a great deal of marking time in grades seven and eight. Practically all of the topics of arithmetic have been introduced by the end of grade six. What is left for grades seven and eight is then a review of processes already taught plus applications to different social situations. To fill the time that remains, most schools introduce some materials from algebra and intuitive geometry. The suggestion made here is that, at least for the brighter third of the pupils, the arithmetic program be terminated at the end of grade six and a suitable program of secondary mathematics be begun in grade seven. While some junior high schools now attempt to do this, their work is handicapped by the fact that their pupils come through elementary schools where arithmetic is conceived as an eight-year program. If arithmetic is to be changed to a six-year program, then some of the topics will have to be introduced sufficiently earlier to allow time for thorough learning and consolidation by the end of the sixth grade. That this is entirely possible is indicated by the evidence that at least in England and in Holland this is done, not only for the top third but for the entire group of elementary pupils. Academic achievement seems to depend considerably on how society values it. Obviously, if such changes in seventh and eighth grade arithmetic were made, it would necessitate a higher level of mathematical preparation for the teachers in those grades.

The basic content of the arithmetic program is rather definitely fixed by the character of the number system and number relations. Even the sequence of topics is somewhat controlled. Essentially, arithmetic requires the mastery of the operation of a subjective number system invented by human intelligence. The possible number relations within this system are determined by the nature of the system. In our decimal system the sum of 15 and 5 must be 20; in a duo-decimal system the sum would be different. Children must learn to count, to carry on the four fundamental operations with whole numbers, fractions, and the special kinds of fractions called decimals. They must understand per cents, ratios, proportions, and the functional relations represented by the facts that if $6=2\times3$, then $6/2=3$ and $6/3=2$. They will then need to learn certain tables of denominate numbers with which the number operations and processes may be used. The number operations and process are quite independent of the societal conditions under which they are used and applied. Eventually, to be of any value, they must be applied to the practical and intellectual problems of society.

The main job of the arithmetic program is to enable children to understand the number system and to become skillful in carrying on

the various operations and processes that are involved. Competent application is possible only if such understanding and skill is acquired. While learning and applying are mingled in the school program, it must be understood that the essence of arithmetic is its fund of mathematical facts and relations. One cannot apply until he has something to apply. The critical lack with both pupils and teachers is in mathematical understanding. I have experimented with college classes repeatedly in learning a number system *not based on ten*, and invariably they react with confusion. Students in educational psychology who can find an *IQ* from the formula $MA/CA = IQ$, are confused if asked to find *MA* when *IQ* and *CA* are given. In more cases than not their difficulty is arithmetical rather than psychological. Recently I asked a group of graduate students to give a quick response to the question $4 \div 2 =$ what? All responded 2. To the next question, $4 \div 1$, they all responded 4. But to the question $4 \div 0$, fifteen students out of a group of twenty-eight responded zero. To make matters worse, half of them wanted to argue that zero was correct. There must be some way to teach arithmetic to bright students so that arithmetical meanings will be clear.

Improving Teaching Materials

If improvements are made in the program of arithmetic, better teaching materials and better teacher-training are necessary. While outstanding teachers can, and do, use textbooks with flexibility, for many teachers the arithmetic that is taught is the arithmetic of the textbook. Obviously, it is easier to say that teaching materials should be improved than it is to deliver pages of actually improved manuscript.

First, one must recognize the limitations of what may be done in any one textbook when applied to an entire class. For the weak pupils the book is too difficult; for the strong it is too easy. Applied exercises cannot possibly be of equal interest to all pupils. Sufficient explanation to make a concept clear to a slow learner bores the rapid learner. One might study with profit the way teachers of reading have turned to a basic book plus multiple supplementary books. This idea might be of use in arithmetic. If a basic book could be built for average pupils it could include all the facts and understandings essential to the subject. This might then be supplemented by booklets which, for the slow learner, would provide the additional explanations and practice materials that would be needed. For the brighter pupils, supplementary booklets of quite a different nature would be necessary. Rather than more explanation and practice, they would need enriching new topics and processes. For example, a booklet on non-decimal number systems might be prepared in attractive and interesting style. It would be for the intellectually curious pupil and it could afford many opportunities for mathematical imagination. Significant applications could be found, such as the use of different bases in computing machines, the use of modified number systems for automobile licenses in some states, the use of indexing systems in libraries, and many others. With travel becoming common, many bright children could be interested in a booklet on foreign money systems and the processes of computing with them. The advantages of a decimal money system might be appreciated better after such a study. A booklet might show the variety of ways in which multiplying and dividing could be done or how answers could be checked. There would be room for a booklet of number puzzles and tricks which would, in turn, contribute to a better understanding of number relations. These illustrations may not be well chosen, but the intent is to find some way to escape from the straight-jacket of uniform textbooks for all types of pupils. Rather than make longer and longer books in attempt to satisfy all users, there might be advantages in shorter basic books with supplementary booklets for special needs. This would permit more flexible programs, especially for the many small schools where sectioning into special classes is impractical.

Improving Teaching Education

Finally, an improved program in arithmetic requires a higher level of teacher training in the content of arithmetic. This is not a criticism of teachers but of the higher institutions that train them. College departments of mathematics and departments of education must in some way find a means to provide advanced work in the subject-matter of arithmetic. In most cases little content is provided that goes beyond that received in the eighth grade. Teachers of science, social studies, English, art, and music all are able to take a college major in their special fields. But arithmetic is not represented in the college program of mathematics, and in most teacher-training institutions more attention is given to methods than to content. Earlier this year I gave to a group of 102 university students, who are planning to be elementary-school teachers, the same 70-item test that was given previously to a large sample of eleven-year-old pupils in England and in California The mean score for the university students was *44* compared with means of *29.1* for English eleven-year olds and *12.1* for California pupils. However, *34* per cent of the prospective teachers made scores below the top third of the English eleven-year olds, and *10* per cent made scores below the mean of the California pupils. These university students are not dull. Their mean score on the American Council Psychological Examination was more than one standard deviation above the national norm for college freshmen. They simply lacked competence in arithmetic. In a few colleges, substantial and scholarly courses in arithmetic are provided, and such courses must become more common.

In the 1951 Yearbook of the National Society for the Study of Education, entitled *The Teaching of Arithmetic*, Grossnickle reported the findings from a nationwide survey of the training of teachers of arithmetic. His data revealed an amazing lack of arithmetical content in courses offered for teachers. In the following chapter of the same yearbook, Professor Newsom outlined what mathematicians think teachers of arithmetic should know. The gap between the training now provided and the training proposed is startling.

This paper has *first* emphasized the need for a better program in arithmetic for those pupils who plan to go on to college, and has urged that the concept of a stretched-out curriculum be reappraised, especially as it applies to the upper third of elementary pupils. *Second*, it proposed that the arithmetic of the kindergarten-primary grades be strengthened and integrated into the total arithmetic program, and also that the arithmetic curriculum for the brighter pupils might be completed by the end of the sixth grade, leaving the seventh and eighth grades for the serious development of high-school mathematics. *Finally*, some suggestions were given about supplementary materials of instruction and about strengthening the program for the preparation of arithmetic teachers.

The main contention of the current critics of the schools is that arithmetic is failing to provide the solid basis of competence that is needed for high school and college programs in mathematics and science. We should give serious consideration to what the critics are saying. However, arithmetic is not taught for the sole purpose of training scientists. It must be organized so that all segments of the school population will be served in the ways that best fit individual needs. Most of what has been written here could have been written with equal emphasis before the first sputnik was launched. Non-scientists also need arithmetic, and, still further, arithmetic is only part of the total enterprise of education. Schools are concerned with the good life to which arithmetic is only one of the contributors. While arithmetic teachers work to improve their own subjects, they must also maintain a sense of balance in respect to the total program of education.

EDITOR'S NOTE. Professor Buswell has presented some well-reasoned observations about arithmetic in our schools. The disturbing things to many of us is that we have so many schools that are not doing what our better schools are doing. The difference probably is due largely to the role of the teacher and her role depends much upon her background of training and her vision of the role of arithmetic in the elementary school program. When we compare results in this country with those in European countries we must remember that we have not given the time nor the attention to excellence of achievement that some other countries have given. We can do much better if we want to do so and if we work at the job. As professor Buswell points out, we should begin with a coherent planned program in the kindergarten and lower grades and not trust to haphazard incidence.

What should we be doing in our school organization so that the better pupils may both "dig deeper" and advance more rapidly? Many years ago we prepared pupils to "skip" a grade. More recently, a number of schools have been using "ungraded" classes. Other shcools are using "flexible grouping" within classes. In a large system it is easily possible to select pupils for a rapidly advancing class and to permit these pupils to complete the normal eight-year curriculum in six years. By properly preparing for "skipping" a grade, a small school can serve its more able pupils.

Are the teachers colleges doing a better job in terms of arithmetic and mathematics than they were a decade ago? Is it still possible for one to become certified to teach who has studied no more than eighth grade arithmetic? Are the people who are "educating" teachers competent to know what constitutes a good program in arithmetic and do they possess the backgrounds for the real education of teachers? What kinds of courses and experiences are needed? Should arithmetic be taught for college credit? Let us re-examine our programs that lead to certification of an elementary school teacher.

Illinois Conference

On April 11th, Illinois State Normal University will be host to the 12th annual conference on "The Teaching of Elementary and Secondary Mathematics." Dr. Hardgrove will discuss "The Subject Matter for Elementary School Mathematics for 1960" and special discussion groups will be organized to deal with problems in the lower and intermediate grades. Lucille Natterstad of Urbana and Mildred Cole will conduct sectional group meetings. Professor Fehr of Teachers College will present "Arithmetic, the Basis of all Mathematics." Francis. R. Brown of Normal will be conference chairman.

ANNOUNCING

UNDERSTANDING ARITHMETIC-Second Edition

Grades 1-8

McSWAIN • ULRICH • COOKE

Consider these proven features:

- Outstanding success in the classroom
- Ease of teaching and learning
- Mathematical correctness
- Abundant opportunities for computational and problem practice

LAIDLAW BROTHERS

RIVER FOREST, ILLINOIS
DALLAS, TEXAS
ATLANTA, GEORGIA
SUMMIT, NEW JERSEY
PALO ALTO, CALIFORNIA

Arithmetic for the Gifted

CYNTHIA PARSONS
Westwood, Mass.

THE TEACHER OF A GROUP of 25 or more active and interested elementary pupils finds herself faced constantly with the problem of what to do with the ones who get through quickly, while she employs her talents and patience with those who need closer supervision.

Some teachers have permitted these "faster" pupils to have a reading book at hand and turn to it so that boredom will not grow into outward undisciplined behavior. While this solution is good for the teacher it may be extremely harmful to the pupil.

A pupil who appears to be good in arithmetic, or who is able to follow the direction of the teacher to satisfactory completion of the assignments, has not proved that he should do less arithmetic. Rather, he has proved that he should be fed more and more. He should be stimulated and motivated to reach to his highest level of achievement—and to do this just as naturally as he maneuvers his bike around town.

Children are no exception to the rule that we all like to do what we known how to do. This can be turned to the busy teacher's advantage. Following are suggestions of activities which are self-motivating and self-directing. All that is necessary is that the teacher get the pupils started and provide the climate and materials for their completion.

In Fourth Grade

Every fourth grade school room should have charts giving the tables of measure: linear, square, liquid. It could be the job of these students to make large charts giving this information.

In pairs or individually, the pupils could first assemble the necessary information. Second, they could work out the details of getting it printed on poster paper for the bulletin board. Third, they could do the actual printing, and illustrating, present it to the class and get it placed in the room.

One fourth grade boy made a whole border of strips of colored paper one inch; twelve inches and thirty-six inches long. It was his own idea.

If the work of the class is on a special step in number work—subtracting with a zero in the subtrahend—then have these pupils work in pairs. They each make up 10 examples of their own, and test the other one on their examples. Let them keep a running scorecard either with a constant opponent, or on a revolving basis. It would be well to collect these individual efforts regularly to give full support and academic weight to the work.

Dominoes and even checkers are games which require arithmetical skill and reasoning. They are certainly quiet, and should call forth no class-room unrest. Let two pupils put their desks together and have a game of either. If the games need the further stimulation, then have contests using the simple ladder system.

If there is sufficient work space, let the children bring scale models (the rather complicated ones requiring real patience and accuracy) to school and work on them there. Or set up some type of model making which will be of benefit to a class-room project, and let these pupils do the more tedious and precise work.

Have them make up thought problems for each other. Write them down clearly. Put the answers on a separate piece of paper, then exchange problems.

They can make up tests for each other which must be completed in just so many minutes. But first the test-maker must try his own test to get par time, then exchange.

When the rest of the class is on Roman

Numerals, these pupils could go to the encyclopaedia for the higher letter equivalents. They could make up simple addition and subtraction examples using Roman Numerals. They could use tag board to make a clock using Roman Numerals. A poster for the bulletin board could be made showing I equals 1; II equals 2; and so on through M equals 1000.

They can get bus or train or plane time schedules and work out departure and arrival schedules and time of travel.

It could be their job to be on the look-out in magazines and papers for numbers over a million. Each one could make his own scrapbook, filling it with articles telling of millions and billions and trillions, etc. Reading them aloud to the class would insure their knowing what they cut out.

I've left one of the most important uses of the children's time for the end. That is, mental arithmetic, or as the children say, "Doing it in your head."

While a certain amount of written number work is necessary to teach orderliness and carefulness, too much of it stifles the "quicker" child. Encourage these pupils, then, to do as much of the work in their heads as possible. In fact, go one step further, once you are sure that they are sure of how to do it written neatly, tell them that they may not write the examples. Even 10 such examples as:

$$\begin{array}{r} 457 \\ \times 2 \\ \hline \end{array}$$

Make them put just the answer down after the number of the example; and then put ten more of the same type (a little harder to do) on the board for them, because they don't have as much writing as the other pupils.

Encourage them to be able to count to 100 by threes as well as by twos and fives.

Encourage them to be able to count backward from 20 as quickly as they can count frontward.

But best of all, pose a complicated problem, have them listen to the facts just once, and then come by the answer by mental arithmetic. For example: a farmer has 20 chickens who lay an average of ten eggs a day. How many dozen eggs do they give him a year? Encourage them then, to make up more of the same for each other.

In Fifth Grade

All the activities listed for fourth graders can, of course, be applied to the fifth grade work. But there are some special techniques and advanced skills peculiar to this grade.

Many fifth grade children have already chosen some special area in which to be an expert. Such as, the naming of all makes of cars simply at sight; reciting the batting average of every player in the Yankee lineup; the maximum and crusing speeds of all Air Force planes; the "fictional" life-history of endless movie stars.

All this can be put to excellent academic use, if the teacher is careful not to let on that it would be "good" to know.

Let these children keep scrapbooks of graphs found in magazines and papers. Then quickly teach them how to construct simple symbol or bar graphs, and let them choose the subject. It might be the heights of ten of their friends, or the batting averages of their All-Star team. The better, more instructive, neater graphs could fill the bulletin board. They could make graphs of their own arithmetic scores; of their spelling; or for that matter a comparative graph for all their different subjects.

November and December are good months to use for the making of calendars. Calendars done in conjunction with the information from the Almanacs. Tidal information could be put on the calendars of those who live in coastal areas. Waxing and waning of the moon, and special historical observations garnered from persual of chronological history source books could also be added.

Stress should be constantly kept on accuracy in the smallest of details. Work which is inaccurate should be re-done, should be re-done as homework, and no student should be allowed the privilege of doing this special

activity unless he is able to be ahead in the regular arithmetic work of the class.

Children this age like to think that they are ready for the *adult* world. Have them collect and bring to school the itemized and totalled slips computed by machine from the large grocery stores. It is a simple matter to cut off the machine-figured total, and mark the slips for later recognition. Then let them do the addition, checking to see if they can come up with the same answer as the adding machine.

Encourage their families to let them keep a running account of the mileage travelled by the family car and the number of gallons of gasoline consumed. Have them get the figures put out by the car manufacturers and check their validity against the individual car used.

The wonderful world of fractions, or parts of things, is opened up now, and these children can be used to help the others in the class. Their construction of fractional facts such as, circles showing a half blackened; a third; a fourth; a fifth, etc., saves the teacher from sleepless nights of "getting together visual aids!"

It might be the job of one child to show the rest of the class positively just how much half a cup of water really is, and how many halves there would in in four cups.

Let them work out the details of just how they put some fact across, of how they will get the materials, of what they will use to prove a point.

To get them started, you might give them all the same task. "Show the difference between one-eighth and one-sixteenth." Let each one present his idea to the rest of the class and have them vote on the one which they thought was the clearest, and thus taught the fact.

Scale drawing is a good activity for teaching the meaning of less than a whole, of ratio, of progression. Start with a scale drawing of a desk, then the school room, then the playground, perhaps the school building, and then on perhaps to part of the town.

To have the children make up problems and examples of their own for their companions to try is excellent for them both. It teaches the one who makes up the problems to think through to the concept of the skill he has learned; forces him to be neat and accurate so that the other will understand what is required; stimulates the imagination to think up something that the other fellow won't think up; and builds up an all important confidence in being able to think arithmetically.

It is possible to explain briefly to these pupils that it is possible to divide first by 8, and then that quotient by 3, instead of dividing by 24.

They should be encouraged to know all the number facts in addition, subtraction, multiplication, and division; know them at sight and give them without hestiation. They can make flash cards from $3'' \times 5''$ index cards and work individually with them. To further encourage this instant recognition, these pupils might perform in front of the others, answering volleys of questions directed to them by the class.

There is no reason why they shouldn't memorize the squares of the numbers through 25. No reason why they couldn't learn the 13 and 14 times tables. Certainly they are ready to grasp that multiplying a factor by \$.25 is the same as taking $\frac{1}{4}$ of the factor.

Have them assemble bulletin board material or make a booklet for the class containing instances of the uses of fractions in the newspapers and magazines.

Conclusion

By employing these and other teaching devices, the classroom is quieter, and the teacher freer to teach. But the real gain is seen in a keener, better balanced individual student.

EDITOR'S NOTE. Miss Parsons has many suggestions for individual and small-group work for the more able pupils in grades four and five. Any teacher should be able to extend this type of work to other grades, both above and below, and she will be able to think of other activities with an arithmetical content. Note that many of these activities may involve little computation but they include thinking, knowledge to be gained, and experience in organization and in presentation. And as Miss Parsons has said, there is a spirit of pride in achievement.

A Comparison of Achievement in Arithmetic in England, California, and St. Paul

THOMAS L. BOGUT
Public Schools, St. Paul, Minn.

FOR THE CONVENIENCE of the reader, the format and much of the information published in Dr. Buswell's presentation has been incorporated in the following article.

When the writer first read Buswell's study in which California children compared unfavorably with English children with respect to arithmetic achievement, several questions arose.[1] Two of these were:

1. Do the results of the California test represent the average arithmetic achievement of the United States?
2. Is the arithmetic program of England actually as superior as the results of this study might indicate?

Buswell cites several reasons for the lower California achievement, such as mobility, race and nationality and it was felt that a second test, representing the more stable and homogeneous Middle West might present a somewhat truer situation. St. Paul, whose children achieve approximately at the mean in the general achievement test in arithmetic administered throughout the nation, was chosen for an additional sample for replication. (See Table I.)

TABLE I

STANFORD ACHIEVEMENT TEST
6TH GRADE MEANS IN READING AND ARITHMETIC
FOR ST. PAUL CHILDREN*

(JANUARY, 1957)

	Reading		Arithmetic	
	Paragraph	Word	Comp.	Reas.
St. Paul Mean	6.8	7.3	6.5	6.9
United States Mean	6.5	6.5	6.5	6.5

* Courtesy of Dr. R. G. Hansen, Assistant Superintendant in charge of Elementary Education, St. Paul Public Schools.

The Test

The writer obtained a copy of the test used by Buswell, together with his permission to reproduce it.[2]

The test is printed in two sections. Section I contains 40 non-verbal computations, of which 31 are evaluated in this study. Section II contains 60 verbal exercises and problems of which 39 are evaluated. Problems involving money terms are not included in the results. The remaining terms were adopted for use in the United States with only minor verbal changes.

The reliability of the examination was .68 for Section I; .87 for Section II; and .89 for the total test, using Kuder-Richardson formula 20. No validity figures were available for the test.

The Sample Tested

St. Paul is divided into ten high school districts. These schools were checked to see how many 5th grades were in the individual high school district. Dividing the total number of 5th grades in the district by four, and even number was obtained in almost every case, and this was the number of classrooms selected to be tested in each high school area. Therefore, each area of the city was proportionately representative of the school population. The classrooms were assumed to have an average of thirty-five pupils. All the schools in each area were assigned a

[1] G. T. Buswell, "Comparison of Achievement in Arithmetic in England and Central California," THE ARITHMETIC TEACHER, Vol. V, February, 1958.

[2] "Arithmetic Test 5" (Test Publication No. 45) of the National Foundation for Research in Education in England and Wales, 79 Wimpole St., London W.I.

number and the required number was drawn randomly. The principals of the selected schools were then asked to participate in the study, and the writer is pleased to report that there was 100% acceptance. Twenty-one schools were used, with a total of 524 pupils. These data are summarized in Table II.

TABLE II

NUMBER OF SCHOOLS AND PUPILS IN THE SAMPLES FROM ENGLAND, CENTRAL CALIFORNIA AND ST. PAUL*

Sample Tested	England	California	St. Paul
Number of Schools	46	39	21
Number of Pupils	2,703	2,483	524

* Urban schools only.

Due to differences in school organization in England and the United States, Buswell kept the age range constant, rather than the grade level. The age level was 10 yr. 8 mo. to 11 yr. 7 mo. as of the month the test was given. This would include the children in St. Paul born in the period from October 1946 to September 1947, inclusive. The writer feels that somewhat truer comparison of achievement between the English and California samples could have been made. Distinguishing between the achievement of the 5th and 6th grades, would in effect compare not only age level achievement, but would enable the reader to obtain a clearer view of number of years in school achievement also. This probably would have given a more accurate comparison of the two countries.

The writer did not feel that there should be a departure from the age of the children in the sample or procedure used in Buswell's study. Therefore, the same procedure was used with one exception. While Buswell used all children of the appropriate age in a selected school, the writer asked the principals to test only a group of approximately 30–36 from any class that fell in this category. The reason is that the writer felt that it might be difficult to obtain permission to disturb from four to five classrooms in a building, which would have been necessary if the entire age group in a school were to be tested. The writer also felt that a group of this number would represent the school almost as accurately as the entire age group. At the time of year that Buswell administered the test (January 1957) most of his sample from the 6th grade.

Most of the St. Paul sample was selected from the 5th grade level. Only 70 pupils were in the 6th grade, and none in the 7th. This is due to the fact that at the time of testing, May 1958, more 5th graders fell into the age group than 6th graders, and there would be fewer problems in administering the test. Consequently, rather than have many classrooms interrupted by testing, one of two classes, depending upon the number of children eligible, were tested in a school.

Due to the variance in age, there were a few schools with two 6th grades that were able to supply the requested number from both rooms without too much difficulty.

Test Administration

The tests were administered by the teachers in a regular classroom situation. Instructions were printed on the test, and a special instruction sheet was included, emphasizing the time requirements. All tests were administered between May 13 and June 3, 1958. After the tests were returned, they were checked again for proper age range. At this point the tests were corrected, and then spot checked systematically and results recounted, to assure correct scoring. The results, as Buswell points out, are merely a body of facts. The additional results from this testing should help in evaluating the comparison of achievement between the two countries.

Comparison of Results

The results of the tests are shown in Tables III and IV. As shown in Table III, the mean scores on the total test (70 comparable items) are 22.6 for the English schools, 12.5 for the California schools, and 19.6 for the St. Paul schools. The means for Section I of the test (computational) are

TABLE III

RAW SCORES ON ARITHMETIC TEST FOR 70 COMPARABLE ITEMS IN RANDOM SAMPLE OF 46 URBAN SCHOOLS IN ENGLAND, 39 URBAN SCHOOLS IN CENTRAL CALIFORNIA, AND 21 SCHOOLS IN ST. PAUL

	England $N=2703$		California $N=2483$		St. Paul $N=524$		Difference E. & C.		Difference St. P. & E.		Difference St. P. & C.	
	Mean	S.D.	Mean	S.D.	Mean	S.D.	Mean	S.D.	Mean	Mean	Mean	S.D.
Sec. I, Computation	14.3	8.8	4.3	2.1	6.5	3.1	−10.0	−6.7	−7.8	−7.8	+2.2	+1.0
Sec. II, Problems	15.3	10.6	8.2	5.2	13.1	6.6	−8.8	−5.4	−2.2	−4.0	+4.9	+1.4
Total Test	29.6	18.8	12.5	6.8	19.6	8.9	−17.1	−12.0	−10.0	−9.9	+7.1	+2.1

14.3 for England, 4.3 for California and 6.5 for St. Paul. The means for Section II (problems) are 15.3, 8.2, and 13.1 respectively. The difference in the means for Section II between England and St. Paul is small and perhaps one reason is that the St. Paul children are above the United States average in their reading ability, which is closely associated with problem solving ability. (See Table II). However, all differences in means were significant at the .01 level using the standard test for significance.

The reader must also keep in mind that although the age level was kept constant, 464 of the 524 pupils tested in St. Paul were in the 5th grade. Table VII reveals that the total means for the 5th grade pupils only is 19.1 and 19.6 for all pupils in the St. Paul sample. The 6th grade mean is 23.1 but the writer feels that because of the small number involved at this grade level, this figure is not a very accurate one. The reader is cautioned to remember that the English children were tested in May of 1955, the end of their school year, and the St. Paul children were tested in May, 1958, also at the end of their school year. The California children were tested in January, which is approximately half way through the school year. Thus, while Table III indicates a fairly large difference in means between California and St. Paul, the difference in months of school is approximately 7 months. California children were tested in approximately the 6.5 grade while St. Paul children were tested in the 5.8 grade. English children were tested in what would roughly correspond to our 6.8 grade. The writer feels that a more valid conjecture might be reached if all three groups had been tested in the same month.

TABLE IV

DISTRIBUTION OF TOTAL SCORES ON 70 ARITHMETIC TEST ITEMS IN ENGLAND, CALIFORNIA AND ST. PAUL

Raw Score	England $M=29.6$	California $M=12.5$	St. Paul $M=19.6$
69–70	7	0	0
66–68	40	0	0
63–65	57	0	0
60–62	73	0	0
57–59	95	0	0
54–56	117	0	0
51–53	99	1	1
48–50	114	2	2
45–47	126	2	4
42–44	110	0	5
39–41	106	0	9
36–38	97	6	11
33–35	116	13	15
30–32	115	27	29
27–29	106	34	32
24–26	142	56	41
21–23	127	111	57
18–20	135	205	71
15–17	152	328	76
12–14	161	443	87
9–11	170	559	43
6–8	197	384	27
3–5	171	218	11
0–2	70	89	3
N	2,703	2,483	524

Table IV shows the distribution of total scores for the three groups. It is interesting to note that St. Paul has the only symmetrical curve of the three. It is also interesting to note that all three means were in the interval above the mode. St. Paul and California were also identical in the achievement of the top pupils. Both areas had only one child in the 51–53 group and two in the 48–50 group. England had 99 and 144 respectively. In the group 38 and above, 1077 English children, 13 California and 21 St. Paul pupils achieved above that point. This

is about 37% for England, .004% for California and 4% for St. Paul. There were thirteen zero papers in England, twenty-three in California, and none in St. Paul.

Table V shows the per cent correct of each item for each country. Of the first four problems (Section I) St. Paul percentages were higher by significant percentages in items Nos. 2, 3, and 4. In Item No. 1, England outscored St. Paul by 1%, but this difference was not significant. A difference of 4% was required for the z score to be significant. In Item No. 19, Section I, the St. Paul children were again outscored by England by 1%. St. Paul outscored California in every item in Section I, by large margins. Section I places more emphasis on denominate numbers than is usual in the United States.

There were five problems in the test which

TABLE V

ITEM ANALYSIS OF 70 ITEMS

Item No.* Sec. I	Per Cent Correct			Item No.* Sec. II	Per Cent Correct		
	$N=456$ England	$N=544$ California	$N=524$ St. Paul		$N=456$ England	$N=544$ California	$N=524$ St. Paul
1	93%	85%	92%	1	91%	89%	95%
2	84	80	91	2	84	85	91
3	76	67	83	3	73	65	84
4	78	59	82	5	59	66	86
5	61	7	25	6	69	55	78
6	70	66	77	9	54	23	43
7	59	4	19	12	46	42	69
8	58	0	9	13	61	15	34
9	64	2	12	15	37	13	12
10	49	3	14	16	48	24	65
11	48	2	13	17	51	19	40
12	57	1	10	18	34	4	24
14	62	4	18	20	43	21	44
16	49	0	4	21	40	7	38
17	57	2	13	22	47	33	45
18	47	1	10	23	52	39	74
19	43	22	42	24	44	28	45
22	45	0	2	25	45	7	25
23	44	1	6	29	47	7	16
25	42	3	8	30	34	5	32
27	37	0	1	32	47	7	27
28	31	0	1	33	45	9	32
30	24	0	1	34	34	7	14
31	26	0	0	36	33	19	36
32	22	0	1	37	31	17	32
33	22	0	0	41	29	4	13
34	17	1	2	42	30	7	20
35	18	0	3	44	21	7	22
36	15	0	0	45	24	3	12
37	14	0	0	46	23	5	22
39	9	4	5	47	22	3	10
				48	25	7	13
				49	12	1	3
				52	20	5	7
				55	11	1	2
				56	21	1	4
				57	13	1	4
				59	9	1	1
				60	3	1	0

* Item numbers correspond to the numbers on the printed test form.

involved hundredweight, a commonly used measure in England. This is not particularly taught in the United States, but is present in the tables of the textbooks. If the hundredweight problems had been removed from the test, the urban English total mean would be 28.0, St. Paul 19.6, and California 12.5. Therefore, it can be seen that there is as much difference between England and St. Paul as there is between St. Paul and California, if the hundredweight problems were removed.

In Section II, items, 1, 2, 3, 5, 6, 12, 16, 20, 24, 36, 37, and 44 were met more successfully by St. Paul children than by English children (See Table VI for sample items). Of the 39 items tested in Section II of the test, St. Paul children outscored English children on 13 items. Again St. Paul children outscored the California sample on all problems except No. 15. In this instance the percentages were 13% and 12% for California children and St. Paul children respectively.

TABLE VI

SAMPLE ITEMS AND PER CENTS OF ENGLISH, CALIFORNIA AND ST. PAUL SAMPLE DOING EACH CORRECTLY

Section One

1. ADD E*—93%; C*—85%; St. P.*—92%
   35
   47
   64
   23
   —
2. SUBTRACT E—84%; C—80%; St. P.—91%
   537
   469
   —
3. MULTIPLY E—76%; C—67%; St. P.—83%
   875
   6
   —
4. DIVIDE E—78%; C—59%; St. P.—82%
   7 | 987
5. ADD E—61%; C—7%; St. P.—25%
   37 ft. 9 in.
   47 7
   38 11
   —
6. SUBTRACT E—70%; C—56% St. P.—77%
   8403
   7596
   —
7. MULTIPLY E—59%; C—4%; St. P.—19%
   7 lb. 4 oz.
   9
   —
8. DIVIDE E—49%; C—0%; St. P.—9%
   8 | 49 tons 1200 lb.
10. SUBTRACT E—49%; C—3%; St. P.—14%
    86 gal. 4 pt.
    47 5
    —
11. MULTIPLY E—84%; C—2%; St. P.—13%
    15 yd. 2 ft.
    7
    —

Section Two

1. E–91%; C–89%; St. P.–95%
   Add together 14, 5, and 9
2. E–84%; C–85%; St. P–91%
   John has 8 books, and I have four times as many. How many have I?
3. E–73%; C–65%; St. P.–84%
   Add 4 to 45 and divide the answer by 7.
5. E–59%; C–66%; St. P.–86%
   Underline the *smallest* fraction here and also write your answer on the line to the right.
   $\frac{1}{3}$ $\frac{1}{10}$ $\frac{1}{5}$ $\frac{1}{6}$ $\frac{1}{4}$ _____
6. E–69%; C–55%; St. P.–78%
   Write in figures the number ten thousand, one hundred and nine.
9. E–54%; C–23%; St. P.–43%
   A train leaves New York at 10 in the morning and arrives at 2 that afternoon. If it travels at 30 miles an hour, what is the length of the journey?
12. E–46%; C–42%; St. P.–69%
    Father was 42 years old in 1951. When was he born?
13. E–61%; C–15%; St. P.–34%
    What is half of 7 1/2?
15. E–37%; C–13%; St. P.–12%
    Write down the number with thirteen hundreds and seven units.
16. E–48%; C–24%; St. P.–65%
    A man rides a bicycle at 15 miles an hour; how long will he take to travel 90 miles at this rate?
17. E–51%; C–19%; St. P.–40%
    How many *minutes* are there between 9:30 A.M. and 10:45 A.M. the same morning?
18. E–34%; C–4%; St. P.–24%
    What is the cost of 1500 pounds of rock at $4 a ton?

* The E, C and St. P. prefixes before the per cents indicate England, California, and St. Paul, respectively.

The St. Paul children tended to fall behind the English pupils in problems such as items 5, 7 and 8, in Section I, which involve the use of denominate numbers. In the actual addition, subtraction, multiplication, and division problems, the St. Paul children were consistently superior to both the English and California pupils.

In summarizing the results, English children ranging in age from 10 yr. 8 mo. to 11 yr. 7 mo. were superior to both the St. Paul and California pupils of the same age range. The St. Paul pupils were superior to those of California. Of the many factors involved which contribute to the achievement of the various groups in this test, some will be pointed out.

TABLE VII

ST. PAUL MEAN SCORES BY GRADE LEVEL
5TH GRADE—454 PUPILS
6TH GRADE— 70 PUPILS

	5th Grade (454)	6th Grade (70)	Total (524)
Total	19.1	23.1	19.6

Factors to Be Studied

A comparison of the different groups cannot be made from the factual results in the tables alone. In attempting to interpret the evidence presented there are several points that must be kept in mind.

I. In a comparison of the groups, the reader must remember that the English start the children one year earlier in school, and by the age of 11 yr. they have had one more year of formal arithmetic training than their American counterparts.

II. Inasmuch as the St. Paul children were exactly one year behind English children in time in school, and seven months time in school behind the California children, allowance must be made for these factors.

III. The reader is asked to note that although the difference in mean raw scores between St. Paul and England is 7.8, most of this difference stems from the results from Section I of the test. The emphasis on denominate numbers influences the total picture greatly. If the reader observes the difference of 2.2 raw score points in the means in Section II between England and St. Paul and remembers that the St. Paul children have one year less of formal instruction than the English children it can easily be seen that the St. Paul children compare favorably with their British counterparts.

In the problem section, the St. Paul children averaging 5.8 yrs. of arithmetic training are only 2.2 raw score points behind the English children who average 6.8 yrs. of arithmetic training. Thus it appears that the St. Paul children in the problem solving area are achieving more per school year than the English pupils. It is interesting to note that the mean for St. Paul for Section II was about twice as high as the mean for Section I. This is also true for the California sample. The English mean for Section II was almost the same as the mean for Section I. It would appear that the English are perhaps placing more emphasis on the denominate number as a part of their arithmetic program rather than the problem solving part. The meaningful approach to arithmetic that is beginning to permeate the United States is probably reflected in the success of problem solving as against the poorer handling of denominate numbers of both St. Paul and California children.

IV. The reader is again asked to observe the fact that the English children did not out-achieve the St. Paul pupils in the simple arithmetic problems. It was only when the problems involved the use of denominate num-

bers that the St. Paul children fell behind. A point of interest is interjected here. The writer was told by several of the teachers who administered the test that, although the children were told not to ask any questions, there were many who asked whether or not they should change the problems to the larger units or leave them in the original form. Although the teachers, in accordance with the instructions, did not answer the questions, they nevertheless complained to the writer about the fact that they did not feel that the test was explicit enough on this point. In fact, one of the teachers with whom the writer had no personal contact went to the trouble of writing a letter expressing basically the same idea. This problem may also have been prevalent in California. If the English children were more accustomed to problems of this type, then it could easily explain the large difference in achievement between England and the United States in problems of this kind. It is conceivable to assume that if several teachers out of 21 made this point, it was not an isolated one, and it might easily have been a complicating factor in many more of the classrooms.

Conclusions

The writer feels that only one valid conclusion can be reached and that is that on the average, English children ages 10 yr. 8 mo. to 11 yr. 7 mo. are superior in arithmetic achievement to American children of the same age. The difference in the means is significant at the .01 level of confidence. However, there are several inferences that we can draw from the body of facts presented.

I. Although the English children achieved higher than the California and St. Paul children, most of this difference resulted from Section I of the test. The results of Section II of the test are much closer, especially between St. Paul and England. The American theory of the meaningful approach is undoubtedly reflected in the Section II scores of the St. Paul and California pupils. It would appear that judging by American standards, the English children are not achieving in the problem solving area equivalent with the denominate problems area. This is probably due to more emphasis on denominate problems.

II. Although this paper assumed that English children had one year more formal arithmetic training than the American sample, in reality it is even slightly more. English children complete in 6 years what American children complete in 8 years. Thus the average English child taking the test would have actually had 1.6 years more formal training than the average St. Paul 5th grader. In this light the differences between the two groups become clearer.

III. Due to the fact that English children are expected to take the "11-year-plus" examinations for entrance to the secondary school, it might well be that they are, on the average, trying harder in the tests than the American children, although this test is not part of the "eleven-plus" examinations. The classes are also grouped by ability and the more able ones, taught by more able teachers, are pushed in order to prepare them for the final examinations. Montague states,

> "It is apparently true that much of the last term of the primary school is given over to 'preparing' the pupils for the eleven-plus."[3]

[3] J. B. Montague, Jr, "Eleven-plus Battle in Education in England," *Clearing House*, 32 (January, 1958) 261.

He also says that

"For months before the examinations, various kinds of pressure are put on the eleven year olds."[4]

This might possibly account for the large upper group among the English children.

IV. The factors of race and mobility probably had a depressing effect on the California scores. There is no doubt that the mobility rate in California is much higher than in St Paul. The race factor in St. Paul was present only in a very minute degree. (10 pupils)

Critical Issues

I. There are many problems involved in a comparative study which are difficult to overcome. In fact as Hollinshead points out,

". . . European aims and methods are so different from ours that the question arises of whether it is even appropriate to try to compare systems which have such different purposes."[5]

II. The difference in philosophy between the two countries can be seen in the statement of Alexander.

"From seven-plus to eleven-plus in the junior stage the work will inevitably become more formalized and the mastery of the three R's is a reasonable expectation."[6]

If this is the case then we must be extremely careful when such a comparison is made and make sure that all factors are weighed before arriving at a decision.

III. The main point under question appears to be whether or not our present organization of the arithmetic program is achieving the desired goals. It is fairly obvious that it is not producing a large number of talent pupils in arithmetic. (See Table IV.) The question then arises, is it necessary in our society to have a large number of talented people in arithmetic? Should the teacher spend more time with the more able pupils? The writer feels that if this were to happen; if we were to devote ourselves to developing a larger groups of talented pupils in arithmetic, then as a logical result the average student the educated follower, might well lose out. The writer feels that we need to develop and educate the followers who will be carrying out the policies of the leaders, as well as to develop leaders. This does not mean that our enrichment program should not be evaluated. More, continued, and better enrichment for the talented children should always be one of our unchanging objectives.

EDITOR'S NOTE. We are grateful to Mr. Bogut not only for his statistical comparison of pupils from St. Paul with those of England and California on this test but also for raising some of the pertinent questions arising from the comparison. Neither this study nor the one by Professor Buswell has given information about the amount of time per day, week, or year that is spent on airthmetic in the schools that are being compared. The large standard deviation of the data for England is due in large measure to the number of pupils who have comparatively high scores and this in turn may be due to the emphasis upon achievement at the stage of the "eleven-plus examinations." It is wholesome for us to take stock of our work and to ponder how and where we are weak and how to strengthen our program. Is there too much emphasis upon denominate numbers in England? Too much emphasis in the test? Is the test a good and valid sampling of the total program and the aims expected to be achieved? In choosing any standardized test one should always study very carefully the content of the test in terms of its range and scope and its representation of a good program.

Mr. Bogut's quotation from J. B. Montague suggests a type of pressure which has been abandoned in most of our schools, a pressure which tended to produce rather high test results which were not lasting. It is very probable that concentrated drill upon the type of exercizes that one expects to find on a test will produce results considerably beyond those normally expected. But that is not a true and valid measure of the effectiveness of a program.

[4] *Ibid.*

[5] Byron S. Hollinshead, "Is European Education Better?" An address at Miami University, Oxford, Ohio, in February, 1958.

[6] W. F. Alexander, *Education in England.* London: Newnes Education Publishing Co. Ltd. 1954 P. 42.

Why Is Arithmetic Not the End?

W. W. Sawyer
Wesleyan University, Middletown, Conn.

In primary school we learn to answer questions like, "What is 3 times 8?" and "If an object cost $8.23, what would 37 such objects cost?" After a bit, someone who was good in arithmetic might be able to add, subtract, multiply and divide any numbers that anybody mentioned. It seems there would be nothing more for such a student to know.

But there is a whole subject that grows out of arithmetic, and yet is not arithmetic. Below we give some questions. Each question is a question in arithmetic. But if you answer these questions, you will notice something about the answers, something you did not expect, something that arithmetic does not explain. And you will be able to guess the answers to much more difficult questions, without doing any working out. (See, for example, the questions in group B.)

To get started, we will do together QUESTION A1. *Work out 0×0, 1×1, 2×2, 3×3, 4×4, and so on. What do you notice about the answers?* The answers are

$$0\times0=0$$
$$1\times1=1$$
$$2\times2=4$$
$$3\times3=9$$
$$4\times4=16$$
$$5\times5=25$$
$$6\times6=36$$

(Of course, we could go on with this as long as we liked.) What do we notice about the answers? Several things perhaps. Someone might notice that 0 is even, 1 is odd, 4 is even, 9 is odd, and so on. We get even and odd numbers by turns. That is quite true. But the thing I mainly hoped you would notice is the way these numbers rise, the change from each number to the next.

From 0 to 1 is an increase of	1
From 1 to 4 is an increase of	3
From 4 to 9	5
From 9 to 16	7
From 16 to 25	9
From 25 to 36	11

We will notice here that we have the odd numbers 1, 3, 5, 7, 9, 11, in order. (What would you expect the next increase to be? Does it give the right result?) You will notice that these numbers, the odd numbers, rise each time by 2.

We might write the whole thing like this

A.1. Our answers	0		1		4		9	
rise by		1		3		5		7
which rise by			2		2		2	

	16		25		36		
9		11					
	2		2				

Here are some more questions of the same kind.

A.2. Work out
0×1
1×2
2×3
3×4
4×5
and so on. What do you notice?

A.3. Work out
0×1
1×3
2×4
3×5
4×6
and so on. What do you notice?

A.4. Work out
0×3
1×4
2×5
3×6
and so on. What do you notice?

A.5. Work out
0×4
1×5
2×6
3×7
and so on. What do you notice?

A.6. *Make up some more questions of this kind for yourself. What do you think will be noticed about the answers?*

Surprising Answers

In the next lot of questions there will also be something to notice about the answers.

Before we get on to that, we must make sure we understand the same thing by the questions. For instance, if I ask, "What is $7-3\times2$?" I expect 1 as the answer. For I work it out, 3×2 is 6 and $7-6$ is 1. You might say, "But couldn't it be 8? For $7-3$ is 4 and 4×2 is 8." Well, it all depends what the question is. 1 is the correct answer to "Multiply 3 by 2, and subtract the answer from 7." 8 is the correct answer to, "Subtract 3 from 7 and *then* multiply by 2."

Mathematicians have agreed among themselves, to avoid this kind of misunderstanding, that, in an expression like $7-3\times2$, the *multiplication is to be carried out before the subtraction.* In the old arithmetic books this used to be emphasized very strongly, but not all the new books mention it.

In Section B, all our questions have multiplication signs and subtraction signs. The multiplications are to be done before the subtractions. Thus, in question B.1, for example, $3\times3-2\times4$ means $9-8$, which is 1. You may only subtract after you have done both the multiplications. You will not get the little surprise if you read the question some other way.

B.1. Work out
$2\times2-1\times3$
$3\times3-2\times4$
$4\times4-3\times5$
$5\times5-4\times6$
$6\times6-5\times7$
and so on. What do you notice?

What do you guess $17\times17-16\times18$ will be? Work it out and see if it is. What do you think $13{,}589\times13{,}589-13{,}588\times13{,}590$ will be?

B.2. Work out
$2\times3-1\times4$
$3\times4-2\times5$
$4\times5-3\times6$
$5\times6-4\times7$
and so on. What do you notice?

What do you guess $86\times87-85\times88$ is? Guess $101\times102-100\times103$ and check by working it out.

B.3. Work out
$2\times4-1\times5$
$3\times5-2\times6$
$4\times6-3\times7$
and so on.

Use the result to guess the answer to a question with large numbers in it.

B.4. Work out
$3\times6-1\times8$
$4\times7-2\times9$
$5\times8-3\times10$

and use this to make some guesses.

B.5. Work out
$4\times6-3\times4$
$3\times7-4\times5$
$4\times8-5\times6$
$5\times9-6\times7$
and so on.

What do you notice about the answers? Is this question like questions B.1 through B.4 or is something different happening now?

B.6. Work out
$3\times8-4\times6$
$4\times9-5\times7$
$5\times10-6\times8$
$6\times11-7\times9$
and so on.

B.7. Work out
$3\times4-1\times5$
$4\times5-2\times6$
$5\times6-3\times7$
$6\times7-4\times8$
and so on.

B.8. Work out
$1\times3-0\times2$
$2\times4-1\times3$
$3\times5-2\times4$
$4\times6-3\times5$
and so on. Anything new here?

B.9. Work out
$1\times1-0\times0$
$2\times2-1\times1$
$3\times3-2\times2$
$4\times4-3\times3$
and so on.

B.10. Something happened in questions B.1 through B.4 that was different from what happened in questions B.5 through B.9. Make up for yourself many questions like the ones above. Sort out those that behave like B.1 through B.4. Can you find out any way of making up questions that will be sure to be like questions B.1 through B.4? Can you find a way of getting questions that will behave like B.5 through B.7? Can you make up questions that will behave like B.8 and B.9?

[*Editor's Note—Page 99*]

Levels of Difficulty in Division

ARDEN K. RUDDELL
University of California, Riverside

DIVISION OF WHOLE NUMBERS is a relatively complicated computational process. It is complicated because it involves each of the other processes: addition, multiplication, and subtraction, and also includes principles unique to division alone. Thorough understanding of division is dependent upon knowledge and understanding of the interrelationships between the four processes—addition, subtraction, multiplication, and division. Following are the understandings commonly made explicit.

"Understandings" in Division

Division is a special case of subtraction. This relationship is most basic in understanding division. For many years leading writers have mentioned this relationship, but teaching procedures which utilize the idea have been limited.

Division is the reverse of multiplication. The combinations involved in each of these processes are the same. However, multiplication consists of combining groups equal in size into one large group while division involves separating a group into smaller parts equal in size. An understanding of these relationships is essential for a complete understanding of the division process.

An understanding of the Hindu-Arabic decimal system of notation and the place value of numbers is essential to understanding the division of whole numbers. Placement of quotient figures and regrouping of integers within the dividend can be most effectively learned when based upon an understanding of value in our decimal system of notation.

Divisor-Dividend-Quotient relationships holds generalizations essential for complete understanding of the division process. As an individual grows in understanding the division process, generalizations should emerge with respect to the relationships between the divisor, dividend, and quotient. Estimating an answer or knowing whether the solution to a division problem is reasonable rests upon an understanding of these relationships. These relationships will be defined more fully in the next section.

Addition-Subtraction-Multiplication within the division process must be understood. A knowledge and understanding of addition, subtraction and multiplication should be developed before approaching division. As the complexity of the division process increases, addition, subtraction and multiplication are used more extensively and become more important to the process.

Understandings of Division Behaviorally Defined

A careful analysis of the division process and all that it involves in working with whole numbers brings to light the aforementioned understandings, concepts, and generalizations. If an individual can quickly and accurately solve a division example it does not necessarily follow that the individual possesses even one of the basic understandings listed above. These understandings probably can best be discovered through careful scrutiny of an individual's written work coupled with a discussion or structured interview.

Listed below are examples of behavior typical of children who possess an understanding of division.*

* Identified as a result of a structured interview of 358 children.

Children's Explanations Show Understanding

1. *Division is a special case of subtraction.*
   1.1 I know there are three twos in six. If I had six blocks, I could take two blocks away at a time until they were all gone and I would take two blocks away three times.
   1.2 I can show you there are four threes in twelve by subtracting.

$$\begin{array}{r} 3\overline{)12} \\ -\ 3 \\ \hline 9 \\ -\ 3 \\ \hline 6 \\ -\ 3 \\ \hline 3 \\ -\ 3 \\ \hline \end{array}$$

I subtracted three four times.

   1.3 I have learned there are eight threes in twenty-four. So I subtract all eight threes at once.

$$\begin{array}{r} 8 \\ 3\overline{)24} \\ -24 \\ \hline \end{array} \quad \text{8 threes}$$

   1.4 I can find out how many sixes there are in forty-two by subtracting. I know there are three sixes in eighteen, so I subtract three sixes all at once. I subtract three sixes again from twenty-four. Then I subtract one six. So I find there are seven sixes in forty-two.

$$\begin{array}{r} 6\overline{)42} \\ -18 \\ \hline 24 \\ -18 \\ \hline 6 \\ -\ 6 \\ \hline \end{array} \quad \begin{array}{l} \text{3 sixes} \\ \\ \text{3 sixes} \\ \\ \text{1 six} \\ \text{7 sixes} \end{array}$$

2. *Division is the reverse of multiplication.*
   2.1 There are nine twos in eighteen because nine times two is eighteen. So I subtract all the twos at the same time.

$$\begin{array}{r} 2\overline{)18} \\ -18 \\ \hline \end{array}$$

   2.2 If I used counting discs I could add three discs four times and get twelve. Or I could start with twelve discs and take away three at a time, four times. So three times four is twelve; there are four threes in twelve and I can subtract them all at once.

$$\begin{array}{r} 3\overline{)12} \\ -12 \\ \hline \end{array}$$

3. *The place value of numbers.*
   3.1 Why does the "1" in the answer go above the "5"?

$$\begin{array}{r} 1 \\ 3\overline{)57} \end{array}$$

      3.1.1 First, you are finding how many threes there are in five tens or fifty.
      3.1.2 Because the "5" is in the tens place and the "1" tells how many threes go in the tens place.
   3.2 What does the "2" mean or stand for?

$$\begin{array}{r} 1 \\ 3\overline{)57} \\ -3 \\ \hline 2 \end{array}$$

      3.2.1 You subtracted three tens from fifty-seven and the "2" means that there are two tens left over.
   3.3 Why do we put the "7" beside the "2"?
      3.3.1 Because you are really subtracting three tens or thirty from fifty-seven, so you have two tens and seven ones left over.

$$\begin{array}{r} 1 \\ 3\overline{)57} \\ -3 \\ \hline 27 \end{array}$$

   3.4 Why do we put the "9" in the answer above the "7"?

$$\begin{array}{r} 19 \\ 3\overline{)57} \\ -3 \\ \hline 27 \\ -27 \\ \hline \end{array}$$

      3.4.1 Well, you have to change the two tens to twenty ones so that you have twenty-seven ones to divide. So the "9" goes in the ones place. Then you subtract all twenty-seven ones at once.

4. *Divisor-Dividend-Quotient relationships.* At the introductory stages, few children grasp these relationships. However, these examples indicate that some understanding is being developed.
   4.1 I know the answer to this problem can't be forty-one because the answer must be smaller than fifteen.

$$\begin{array}{r} 41 \\ 3\overline{)15} \end{array}$$

   4.2 Two fourth-grade children out of 358 responded to this question:
   How many sixes are there in 42?

$$6\overline{)42}$$

I can figure there are fourteen threes in forty-two. When the divisor is twice as big, the quotient is half as big. So there must be seven sixes in forty-two.

Other examples could be cited and perhaps other understandings might be selected as basic to the division process. For purposes of this discussion, however, the selected understandings defined above embrace the situations which might arise in a division example. It was the purpose of this article to show that many concepts, principles, and generalizations are basic to division and to define these understandings in behavioral terms.

EDITOR'S NOTE. We all agree that division is difficult and that it requires a high order of thinking. A casual observation of adults, generally, will show that few of them are proficient at division involving more than one digit in the divisor. But many many pupils have learned the process rather well and hence we must conclude that it can be learned. Dr. Ruddell has listed understandings involved in the development of division and through interviews with pupils has shown how children explain in terms of the items of understanding. This procedure can show how well children comprehend the nature of the work they are doing. It may be more worthwhile at certain stages to spend time on learning the "thinking" than on learning the procedures with numbers in the conventional computation.

Editor's Note—Sawyer "Why Is Arithmetic not the End?"

EDITOR'S NOTE. Arithmetic can be much more than getting answers to computations and problems. It can involve thinking and discovery of mathematical ideas and principles. Dr. Sawyer has arranged several simple sequences which suggest patterns which pupils may discover. Try these with your pupils in grade six and above. Here you will find which pupils have gained a modicum of insight. Can they generalize and predict in terms of their discoveries? Many different configurations of digits result in interesting patterns. Perhaps your pupils will develop some of these. The digit 9 is interesting. It is frequently used in "casting out 9" in checks of computations. But it occurs singly or in multiple in many other arrays which may be used in framing puzzles. Let some of your better pupils "play with numbers and ideas" and see if discovery results. These simple discoveries may well be the initial steps toward mathematical insight.

Satisfying the current need

for improved arithmetic

teaching GROWTH IN Arithmetic *Revised Edition*

stresses the "big ideas" in arithmetic—the structure of arithmetic—the fundamental concepts required in arithmetic reasoning.

BY CLARK · JUNGE · MOSER · SMITH

World Book Company Yonkers-on-Hudson, New York

Chicago, Boston, Atlanta, Dallas, Berkeley

Children's Success with Two Methods of Estimating the Quotient Figure

FRANCES FLOURNOY*
University of Texas, Austin

MOST OF THE RESEARCH which has been done regarding methods of estimating the quotient figure when dividing by a two-place number has attempted to show which method will correctly yield the true quotient most often on the first trial. Children's success with any method for estimating the quotient figure also depends on their being able to remember and apply correctly the rule for the method which they are taught.

In order to estimate the quotient figure in the example, $26\overline{)159}$, the person using the *apparent quotient method* would think, "How many 2's are in 15 or how mnay 2 tens are in 15 tens?" Since 25 is rounded to 20 or 2 tens for purposes of estimating the quotient figure, some refer to this method as the round-down method. The person using the *increase-by-one procedure* for estimating the quotient figure in this same example would think, "How many 3's are in 15 or how many 3 tens are in 15 tens?" In order to aid in the selection of the trial quotient figure, the 2 tens are increased by one ten to 3 tens because 26 is nearer to 30 or 3 tens than it is to 20 or 2 tens. This latter method is sometimes referred to as the *round-up method*. In this particular example, the trial quotient is not the true quotient with either method.

Those using the *increase-by-one* or *round-up method* have usually done so only with the examples which have 6, 7, 8, or 9 in the one's place of the divisor, and they use the *apparent* or *round-down method* with divisors having 1, 2, 3, 4 or 5 in the one's place. In other words, they round both ways depending on the size of the figure in the one's place of the divisor. Because this procedure involves the use of two rules, it is sometimes called the *two-rule method*. Those who use the apparent or round-down method in all cases use a *one-rule method*.

There seems to have been only one major published study concerning a comparison of pupil success with the two different methods of estimating the quotient figure as described above. About twenty years ago, Grossnickle[1] made a study in an attempt to determine the relative merits of these two methods for estimating the quotient figure. Two groups of fourth grade children were used in the experiment. One group was taught the *round-down method* and the other group was taught the *round-both ways method*. Practice exercises planned by the experimenter were carried out by both groups for a period of 76 days. Grossnickle found no significant difference between the mean scores on a final test taken by the two groups of children.

Points for Consideration

Those who favor the *round-both ways procedure*, in which the direction for rounding depends on the size of the number in the unit's place of the divisor, seem to base their argument for this method entirely on the fact that it most often yields the true quotient on the first trial. Those who do not favor a *round-both ways procedure* base their argument on certain features of it which are termed undesirable. Because of these features those in the latter group believe the *round-both ways method* to be more difficult for the child to apply correctly when estimating the quotient figure with a two-place divisor than is the *round-down method*.

* The writer gratefully acknowledges the cooperation of Mary Ann Edwards, M. K. Hage, Frances Osborne, and Corrine Williams, fifth grade teachers in Oasis School, Austin, Texas.

[1] Foster E. Grossnickle. "An Experiment with Two Methods of Estimation of the Quotient," *Elementary School Journal*, Vol. 37, (May, 1937), 668–77.

The undesirable features of the *round-both ways method* are pointed out by those in the latter group to be the following:

1. This method requires the learning of two rules for estimating the quotient figure.
2. In order to make a correction in the first trial quotient the child must sometimes decrease the trial quotient and at other times must increase it. With the *one-rule* or *round-down method*, the correction is always made by decreasing the trial quotient figure.
3. It is believed more difficult to see the need for correction when the divisor has been rounded up in order to estimate the quotient figure. In A below the *round-down method* has been used and the need for correction is seen as soon as the child has multiplied the trial quotient by the divisor. It is not possible to subtract.

A	B
7	511
26)1593	26)1593
182	130
	29
	26
	33
	26
	7

In B, the *round-up method* has been used. If a child failed to compare the remainder with the divisor, he might make the kind of error shown in B. With the *round-up method* the need for correction is not recognizable until the subtraction has been performed.

On the basis of the arguments for and against the *round-both ways method*, the following questions are being considered in this article:

1. Do children who are taught a *round-both ways method* actually apply both rules?
2. When dividing by a two-place divisor, to what degree is the first error an error in choice of quotient figure?
3. Do children who use a *round-up method*, with divisors having 6 or more in the one's place, fail to see the need for correction in a case like that shown below and try to proceed with the division without making the necessary increase in the trial quotient figure?

$$\begin{array}{r} 4 \\ 28\overline{)1436} \\ 112 \\ \hline 31 \end{array}$$

4. Do children who are taught a one-rule procedure for making all corrections in the quotient figure make this correction with greater accuracy than children who have learned two rules for correcting the quotient figure?
5. How do low achievers in arithmetic computation who are taught and actually apply the *two-rule method* compare in performance with low achievers who have an opportunity to learn the *two-rule method* but choose to use the *one-rule method*?
6. Do teachers who teach a *two-rule method* consider certain features of it to be undesirable?

Procedure

The cooperation of four fifth grade teachers was secured in order to consider the questions raised above. Two of the teachers agreed to use the *one-rule method* of rounding down for teaching children to estimate the quotient figure when dividing by a two-place number. The other two teachers agreed to teach the *two-rule method* of rounding both ways. In the two classes using the *one-rule method*, the arithmetic grade equivalent average in computation for the 61 children on a standardized test given in October, 1956 was 5.4. In the two classes using the *two-rule method*, the arithmetic computation grade equivalent average for the 63 children was 5.1.

No restrictions were placed on the teaching procedure used by each teacher and the amount of practice provided. This was left to the judgment of the teacher according to the needs of the children being taught. Since this article is not concerned with trying to offer statistical data which might prove one method to be more effective then another, this plan was considered to be satisfactory for purposes of exploring the questions being considered.

During the last week in October, 1956, each of the four teachers introduced division by a two-place divisor. Each teacher taught the *round-down method* at first and used only those examples in which the one's digit in the two figure divisor was 5 or less. After, in each teacher's judgment, sufficient practice had been given on examples of this kind, examples with 6 or more in the one's place of the divisor were used. At this point, two of the teachers continued teaching the *round-down method* for estimating the quotient figure and the other two teachers taught the *round-up method* for use with examples having 6 or more in the one's place of the divisor.

During the second week in April, 1957, a test, constructed by the writer and consisting of ten examples involving two-place divisors, was administered by each teacher. Five of the examples involved divisors with 5 or less in the one's place of the divisor, and five of the examples had 6 or more in the one's place of the divisor. The children were instructed not to use any method of checking after solving each example. There was no time limit on the test so that every child was given time to finish. In order to know which method for estimating the quotient figure was being used by each child, each was directed to write in his own words how he thought in deciding on the first quotient figure in the last two examples on the test. One of these examples involved the divisor, 73, and in the other example the divisor was 27.

An analysis of the content of the ten test examples reveals that the children using the *one-rule method* had to make seven corrections in the first estimated quotient figures by decreasing the trial quotient by one each time a correction was needed. The children using the *two-rule method* had to make six corrections in the first estimated quotient figure. For these children it was necessary to increase the trial quotient by one to make three of the corrections and to decrease the trial quotient by one in making the other three corrections.

Findings

Through visiting in the classrooms, interviewing the teachers, and studying pupil responses on the test administered, the writer was able to make certain observations regarding the questions being considered in this article.

1. Of the 63 children who were taught the *two-rule method*, 44 seem to have applied both rules, whenever each was appropriate, on the test. Seventeen children used only the *round-down method* for all cases on the test according to their description of how they thought in selecting the first quotient figure in the last two examples. Two children seem to have been using a *round-up method* for all cases.

   Eleven of the seventeen children using the *round-down method* in all cases were below average achievers in arithmetic computation on the standardized test given in October. Of the two who seem to have rounded up in all cases, one child was an average achiever on the computation test but did not perform well on the ten-example division test. The other child was slightly above average on the standardized test and did perform well on the division test.

   When the second rule of the *two-rule method* was introduced to the children in one of the classrooms, this writer observed in the classroom. It appeared that the teacher was successfully helping the children understand and see advantages in the using a second rule of rounding up the divisor. However, when three groups of children with five children in each group worked at the chalkboard, there seemed to be only one child in each group who was consistently applying the second rule when appropriate. Others seemed to be continuing the use of the first rule of rounding down for all cases. In each group after the children had solved about three examples followed by some discussion of how they thought in selecting the first quotient figure, one or two other children in each group began applying the second rule.

2. Every incorrect example for every child was examined to note the kind of error made at the point of first error. Among the 63 children who were taught the *round-down method* there were eight first errors that were quotient selection errors when the one's digit in the divisor was 6 or more. There were also ten other first errors among these children which were quotient selection errors when the one's digit in the divisor was 5 or less.

   Among the 44 children who used the *round-up method* when the one's digit in the divisor was 6 or more, three of the first errors made were quotient selection errors. There were also four other first errors among these children which were quotient selection errors when the one's digit in the divisor was five or less.

   In all except one of these first errors occurring in the choice of a quotient figure, the child making the error would probably have caught and corrected his error if he had multiplied or subtracted correctly. The fact remains, however, that the wrong choice of the quotient figure played some part in the cause of error in the division example. There were 53 situations in which the first error in the division example occurred in subtraction. In 58 situations the first error occurred in multiplication. Errors in subtraction and multiplication caused more than four times as many first errors as did errors in choice of quotient figure.

3. Only one child among the 44 using the round-up method failed to see the need for correction in a situation causing an error such as that shown below. The belief that such a need for correction is hard to see has been stated by some as being a disadvantage

```
   77
56)4382
  392
  462
  392
   70
```

in the *round-up method.*

Three children among the 63 taught the *round-down method* decreased too far in making the necessary correction in the second estimated quotient figure for this example. Errors like those shown below were found.

```
   711          76
56)4382      56)4382
  392          392
  462          462
  392          336
   70           36
   56
   14
```

4. In order to observe which group made corrections in the estimated quotient figure with greater accuracy, these examples were examined in which children in both groups had to decrease by one in order to make the necessary correction. Of the 14 first errors made which were quotient selection errors when the divisor had 5 or less in the one's place, nine of these occurred in examples where a correction was necessary because the first trial quotient was incorrect. Six of the children learning the one-rule method failed to make the required correction of this kind. Three of the children in *two-rule method* group failed to make the appropriate correction. In making a necessary correction in the estimated quotient figure when the divisor had 6 or more in the one's place of the divisor, the *round-up rule* group was again more accurate.
5. It has been mentioned that eleven of the low achievers in arithmetic computation, as achievement was indicated by a standardized test given in October and who were taught the *two-rule method*, seemed not to use the *two-rule method* for estimating quotient figures on the division test. Instead they seem to have used only the one-rule of rounding down for all cases. These eleven low achievers had a computation grade achievement of 4.6 on the standardized test and an average of 2.2 examples wrong on the division test.

   The ten low achievers who seemed to apply both rules for estimating the quotient figure on the division test had a 4.5 computation achievement average on the standardized test and an average of 1.7 examples wrong on the division test.
6. The two teachers who taught the *two-rule method* for estimating the quotient figure agreed to teach this method because as teachers they preferred to teach it. They also had experience in teaching only the *one-rule method.* Part of their preference for teaching the *two-rule method* may lie in the fact that this is the method they, as individuals, use for deciding on the quotient figure. In addition, they indicated preference for the *two-rule method* because of the greater ease and speed it provided in selecting a quotient figure. As teachers, they did not consider this method to have any greater possibilities for pupil errors than does the *one-rule method.*

   The two teachers who agreed to teach the *one-rule method* personally used this method and also preferred a method involving the teaching of only one rule.

Conclusions and Comments

1. Some of the children who had an opportunity to learn the *two-rule procedure* for estimating the quotient figure did not actually apply both rules on the test. A number of low achievers in arithmetic continued to use the *round-down method* only.

   Low achievers may have difficulty in learning two rules for estimating the quotient figure. Some may feel more secure in the use of only one rule. However, the low achievers in the group having an opportunity to learn both rules and who actually applied both rules had slightly better test results than those who had this opportunity also but only applied the *one-rule procedure.*
2. Failure to recognize a need for making a correction in the quotient figure was no more prevalent among those who learned a *two-rule procedure* than it was among those who learned a *one-rule method* of always decreasing to make a correction. In fact, the failure to recognize a need for correction was not a common cause of error.

   There was error, however, in actually making a correction in the trial quotient once the need for such a correction was recognized. There was some slight indication of a tendency to over-decrease among those who were taught the decrease method of making all corrections in the estimated quotient figure. A tendency to over-increase was not observed among those who sometimes needed to increase in order to make a correction in the estimated quotient figure.
3. If a child divides correctly, places the quotient figures correctly, multiplies accurately, subtracts, without error, and checks his remainder with his divisor to see that it is smaller, there is practically no possibility of error in selecting the quotient figure. On the test used in this investigation, children made frequent errors, however, with all of these aspects of the complicated process of division by a two-place number. Thus, errors in computation more often caused the wrong choices of quotient figure than did inability to remember and apply correctly the rule taught for estimating the quotient figure.
4. This analysis of children's errors did not reveal that certain features of the *round-both ways method* make it any more difficult to apply correctly than is the *round-down method.*
5. The fact that the *two-rule method* yields the true quotient most often on the first trial is an advan-

tage with which one can not argue. It seems that children should have an opportunity to learn the *two-rule method* if they can do so without making errors.

From observations made in this investigation it does not seem wise to *require* all children to apply a *two-rule procedure* for estimating the quotient figure. As has been stated some of the children who learned a *two-rule method* actually applied only the *round-down rule* and were generally successful. It cannot be known whether they would have been as successful if they had been *required* to apply the *two-rule procedure*. However, those children who actually applied the two rules in this investigation were generally very successful. It may be that at some stage in their development all children should have an opportunity to learn the *two-rule procedure* and then to develop skill with the method with which they are most successful.

6. As was pointed out earlier, there seems to have been only one major piece of published research which compares the performance of two groups of children using the two different methods as considered in this investigation for estimating the quotient figure when dividing by a two-place number. No significant difference in performance was found.

   In the study previously cited there was no evidence presented to prove that all of the children who learned a *two-rule method* actually applied both rules. Further research with attention to this factor might reveal new and different evidence as to the merits of the *round-down* and *round-both ways methods* of estimating the quotient figure. The comparison of children's performance on a timed test and an untimed test might also reveal a different kind of evidence.
7. In addition, a carefully controlled study of the kinds of errors children make when using the two methods of estimating the quotient figure should provide a new kind of evidence as to children's chances for success with each of these methods.

EDITOR'S NOTE. "Long Division" has long been considered the most difficult procedure learned in the elementary school. It is not a simple one-step process. To become proficient in long division with divisors of two or more digits requires a considerable amount of thinking and some of this is of a fairly high order. Many people have tried to make the process more mechanical and thereby hoped that children could become proficient with greater ease. Finding the correct quotient figure and doing this with ease and surety has been the subject of investigation. Now Dr. Flournoy has sought further evidence in the controversy of "round up," "round down," or "round both ways." She has not attempted to carry on a controlled experiment but rather she wishes to shed more light on what children actually do when taught a specific method. Read her conclusions and see if you may wish to revise your thinking about this problem. It would appear that different pupils probably should think and perform differently and that the level of insight that a pupil uses seems to depend upon both his mental capacity and the ways in which he has been taught. Let us not limit our thinking about division to consideration of ways of improving our conventional method. We need more experimental work on some of the less-direct methods that feature insight into mathematical relationships. And we need to give more attention to levels of work and levels of achievement for different types of pupils.

The National Council's Committee on Arithmetic

A Report of Progress to Date

THE BOARD OF DIRECTORS of the National Council of Teachers of Mathematics has charged the Elementary-School Curriculum Committee (ESCC) with the following responsibility:

> "To develop plans for a study of the elementary school curriculum in mathematics and to propose methods for securing the needed financial support."

At the outset of their work in relation to this responsibility the members of the ESCC clearly recognized the need for special emphasis upon the *mathematical content* of the arithmetic curriculum: its nature, scope, organization, sequence, and the like. We also clearly recognized that to be most effective, such a curricular study should not be designed and executed from the standpoint of the elementary school alone. Rather, it should take cognizance of and be related to significant work and developments that have been and are taking place in regard to the mathematics curriculum of the secondary school, at both the junior high school and senior high school levels.

Consequently, the members of the ESCC

first devoted considerable time to a consideration of the work of groups such as the following, with particular attention to implications for the mathematics program at the elementary-school level:

1. The Secondary-School Curriculum Committee of the National Council of Teachers of Mathematics.
2. The Commission on Mathematics of the College Entrance Examination Board.
3. The UICSM Mathematics Project, and the newly initiated University of Illinois Arithmetic Project.
4. The University of Maryland Mathematics Project: Jr. H. S.
5. The School Mathematics Study Group.

We also took into account things such as:

6. The relevant sections of two future NCTM Yearbooks: the 24th on *Mathematical Concepts* and the 25th on *Arithmetic*.
7. Significant elementary-school mathematics curriculum projects recently completed, or currently in progress, at local and state levels.

Out of this background the ESCC formulated proposals relating to a comprehensive study of the elementary-school mathematics curriculum that embrace the following complementary phases of activity which would operate more or less simultaneously, each contributing to the progressive development of the other:

A. "*Normative*" *Research* and analysis of the current status of the elementary-school curriculum in mathematics, with special emphasis upon content considerations. This analysis should take into account the objectives of mathematics instruction at all levels, and should reveal the significant strengths and weaknesses of current status at the elementary-school level in relation thereto.

B. "*Action*" *Research:* the development and preparation of teaching-learning materials (i.e., "units," "sequences," and the like) relating to selected phases or aspects of content appropriate for the elementary-school curriculum in mathematics; the try-out of such instructional materials in actual classroom situations; the subsequent revision and refinement of these materials.

Both phases of research may suggest needed controlled experimental studies. In any event, the proposed study would lead to the progressive formulation of conclusions and recommendations regarding the elementary-school mathematics curriculum,—the nature, scope, organization and sequence of its content,—from the standpoint of the elementary-school itself, and from the standpoind of the relation of the elementary school to the junior and senior high schools.

At the recent Christmas Meeting of the National Council of Teachers of Mathematics (held in New York City) the Board of Directors approved the ESCC's research recommendations, along with plans for setting up a suitable preliminary operational organization in order to initiate and maintain the proposed study in direct association with an ongoing mathematics curriculum project at the secondary-school level.

Subsequent issues of THE ARITHMETIC TEACHER and *The Mathematics Teacher* will carry further progress reports from the Elementary-School Curriculum Committee as plans materialize.

Joyce Benbrook: University of Houston
Laura K. Eads: Bureau of Curriculum Research, New York City Public Schools
Ann C. Peters: State Teachers College, Keene, N. H.
Irene Sauble: Division of Instruction, Detroit Public Schools
Henry Van Engen: University of Wisconsin
J. Fred Weaver (Chairman): Boston University

Annual Business Meeting

Notice is hereby given, as required by the Bylaws, that the Annual Business Meeting of the National Council of Teachers of Mathematics will be held at the Baker Hotel, Dallas, Texas, at 4:00 P.M., Friday, April 3, 1959.

Helping Children Understand Verbal Problems

MILO K. BLECHA
University of Arizona, Tucson

IT DOESN'T TAKE ANY RESEARCH study or report to tell either teachers or parents that many children have a great deal of difficulty with verbal problems—also referred to as work problems, sentence problems, or story problems. This, of course, is no new or startling situation. Most of us can recall going through the same struggle with these problems as children do today. In fact, many of us still wouldn't do too well with some of the verbal problems in a sixth or seventh grade arithmetic book.

The question that naturally comes to mind at this point is: why do children have more difficulty with verbal problems than other mathematical problems? Part of the answer lies in the nature of the problems, because verbal problems involve more than other arithmetic problems. The individual must first read the problem, and then decide which process or processes to use in order to solve it. The challenge immediately presented is not one of manipulating numbers, but rather, one of deciding whether to add, subtract, multiply or divide? The following statement puts it very well.

> "the pupil must not only be able to understand the vocabulary of the statements but he must also be able to visualize the situation that is presented and to sense the relationships among the quantitative elements that are involved. In addition to this he must be able to perform the necessary computations to find the answer to the problem."[1]

In verbal problems, there are no signs to indicate how to solve them. In many problems, the signs give a sense of security that so often is lacking in the verbal ones. Thus we see why children become confused and even afraid of story problems. Their state of mind and apprehension is reflected in their complaints about the problems, which usually are:

> I don't understand them.
> I don't know what to do.
> It doesn't tell whether to add, subtract, multiply or divide.
> I don't know how many of the numbers in the sentences to use.
> I don't know what some of the words mean.

Their insecurity in dealing with work problems soon leads to such a strong dislike that a psychological barrier is raised which must be overcome before the child can achieve any degree of mastery.

Many studies have been made to determine the causes of these difficulties in the solution of verbal problems. Standard reference books as well as research summaries. List the difficulties and these are about the same as any good teacher has discovered. There are two basic factors which determine a child's ability to cope with word problems.

The first is his mental ability, which the teacher, of course, is not in a position to control. The second factor, however, is concerned with the richness and thoroughness of his background in meanings, understandings, concepts and skills; these can be very definitely influenced by the teacher and deserve special attention.

Part of the difficulty that children have with verbal problems is due to the fact that the child does not know certain words and/or their meanings in the problem. This is often the case with poor readers. However, there is more involved than the problem of reading. It seems there are certain specialized reading skills required for solving verbal problems, and these must be developed through direct instruction. These skills do not, we now know, result as a by-product of the regular reading program. It should also

[1] Brueckner, Leo J. and Grossnickle, Foster E. *Making Arithmetic Meaningful*, Winston Publishing Co., Philadelphia, 1953, p. 492.

be obvious that they cannot grow out of practice in regular arithmetic computation. Therefore, the teacher must work for such reading skills as locating information, reading for detail, organizing factual data and remembering what is read. Many children, when given verbal problems, skim through the reading to find the numbers given and then begin using the process that first comes to mind, which is usually the one that they have studied most recently.

Another consideration in this entire problem is the fact that arithmetic has a vocabulary all its own. Many of the technical terms used in arithmetic do not appear even once in the child's reading textbooks. Then special vocabulary exercises must be used along with practice in computation to insures the child's comprehension of the language of mathematics. The procedure in teaching this specialized vocabulary is very much like developing vocabulary in other areas of the elementary school curriculum. What is needed is simply a realization of the need for this type of instruction and then a provision of time for it.

Of course, developing the necessary reading skills and vocabulary is not enough to enable children to master work problems in itself. Verbal problems do not constitute a separate, isolable division of the arithmetic program, but rather are an integral part of the whole. They both take from and contribute to the whole program. How well prepared children are to attack work problems depends on how good their arithmetic background is. While this should be a self-evident fact, we sometimes forget it and try to "treat" just verbal problems, hoping they will improve.

With this general background, let us move on to a discussion of more specific methods. Assuming we have done all we could to give the child a good background in arithmetic and to find realistic problems with which to challenge him, what do we do when we reach the verbal problems? Below is a list of steps which may guide the teacher in making a logical plan on attack on verbal problems:

1. Help the child to view the problem as a *sensible* one. The problem should be sensible.
2. Get the child to re-state the problem with understanding and meaning.
3. Decide on the information that is needed in solving the problem and how this information is to be organized.
4. Determine the skills and fundamental processes necessary to the problem's solution.
5. Decide, after solving and arriving at an answer, if the answer is reasonable and verify it with some previous known facts.

While the steps mentioned above are given to illustrate a plan which might facilitate the learning process in solving word problems, it can also be of some value to the teacher in planning her instruction in arithmetic. Most of them are self-explanatory and need no elaboration.

Finally, it must be emphasized that there are certain principles which every arithmetic teacher should bear in mind in working with children in this area. These are:

1. Ability to solve word problems must be a concern of all teachers in the elementary school and it must form as vital a part of the first grade as it does later ones.
2. Ability in working with word problems comes through a variety of well-chosen experiences and grows slowly; it cannot be hurried.
3. Children must learn a variety of sound language patterns and not depend upon a few specific cues of limited and perhaps doubtful usefulness.
4. The learner must develop an understanding of the meaning of the operation and the interpretation of the problems in the light of these operations and their social significance in school as well as outside of school.

Each of these factors is equally important. Without the complete and thorough development of all four, children will continue to encounter difficulty in problem solving arising from verbal problems.

EDITOR'S NOTE. Yes, problem solving in arithmetic is difficult. It requires thinking, understanding, computational ability, and practice. Dr. Van Engen has suggested a study of "structure" as a key to the development of problem solving ability. Pupils need to understand the essential structural relationships expressed by words. There are many aids toward this end. One that many teachers find worthwhile is to have children compose problems from a situation with which they are familar. A study of situations can lead to an understanding of the role of computation and likewise a study of the uses of computations can help in understanding the essential relationships in a mathematical situation.

Sources of Articles Published in the Arithmetic Teacher

AGNES G. GUNDERSON
Moorhead, Minn.

In reading the Report of the Membership Committee by Mary C. Rogers which appear periodically in THE ARITHMETIC TEACHER, one is impressed with the membership growth in the various regions of our country. The question arises: To what extent are the different geographical regions contributing articles to THE ARITHMETIC TEACHER?

A canvass was made of the articles which have appeared in THE ARITHMETIC TEACHER from its beginning, February 1954, to and including April 1958 to determine the contribution made by the different states. The following table summarizes the findings.

In its first four and a half years, THE ARITHMETIC TEACHER has published 221 articles and thirty-eight of these have been reports on research. Thirty-five different states and the District of Columbia plus several foreign countries have been represented.

EDITOR'S NOTE. Each year since 1954 THE ARITHMETIC TEACHER has increased in size and in circulation. The editors want to make this a most useful *Journal* to serve the development of arithmetic in the schools. To this end articles of various types are sought. The editors are constantly seeking articles that describe some good practice in the schools, a procedure that a teacher has found particularly effective. Many favorable reports about the *Journal* have been received and these encourage the editors to try to do a better job. But good articles depend upon the work of people in the field. Let us continue to improve the magazine, let us have your contributions.

TABLE I

Name of State	Number of Research[1] Articles	Total Number of Articles	Name of State	Number of Research[1] Articles	Total Number of Articles
Alabama		3	Missouri		6
Arizona		1	Mississippi	1	2
Arkansas		1	Montana		1
California	6	29	Nebraska	1	4
Colorado		2	New Hampshire		1
Connecticut		5	New Jersey	0.5*	12.5*
Dist. of Columbia		2	New York	7.5*	41.5*
Florida		4	North Carolina	1	3
Georgia		1	Ohio	1	6
Idaho		1	Oklahoma		2
Illinois	4	13	Oregon		1
Indiana		6	Pennsylvania	1	10
Iowa	2.5*	8.5*	Tennessee		1
Kansas		4	Texas	2.5*	8.5*
Maryland		2	Washington		4
Massachusetts	3	9	West Virginia		1
Michigan		12	Wisconsin	3	6
Minnesota	4	6	Wyoming		1

[1] Research articles are those classified as such by Dr. Weaver in the April 1957 and April 1958 issues of THE ARITHMETIC TEACHER and includes none published later than December, 1957.

* One of the articles was written jointly by persons from two different states. Each of the two states is given credit for half the article, e.g. an article written by one person from Iowa and another person from Texas, each state is credited with 0.5 of the article.

What about Those 'Zero Facts'?

ROBERT A. OESTERLE
Purdue University, Lafayette, Ind.

IT IS, in a sense, paradoxical that zero, the discovery of which Dantzig[1] characterized as "one of the greatest single achievements of the human race," should provide such fertile ground for controversy in the methodology of teaching arithmetic in the elementary school.

Brumfiel[2] in THE ARITHMETIC TEACHER observed that there should be no difficulty in teaching the concept of a null or empty set in addition and multiplication. Experience would indicate that teachers are able to 'teach' and children can 'learn' the *zero facts* in multiplication and addition, but the question arises as to the wisdom of presenting these facts as definitions before the need for them occurs.

Utilizing the questionable synonymy of 'zero' and 'nothing,' Wheat[3] stated that a pupil has no "need for a sign for 'nothing.' If he wishes to write nothing, he writes nothing; that is, he does not write anything." In another section of his text, this writer declared:

> It often appears to teachers that the children will have to add, subtract, multiply, and divide zero just as they add, subtract, multiply, and divide when the nine numerals are used. So, as a means of preparation, the 'zero combinations' are 'taught.' There is much explaining and more drills. Through persistent effort, the children finally learn to set down the approved answers when the zero is used. Since the actual use and meaning of the zero are neglected, what the children learn to do is necessarily barren in meaning. Their human minds are turned into machines for use in responding to the 'zero combinations' by giving what the teacher says are the correct answers.[4]

Brueckner and Grossnickle[5] stated that "zero facts in multiplication [should] be presented as a group in two-place numbers," the inference being that teaching the *zero facts* should be delayed until problems of the type $\begin{array}{r} 40 \\ \times\ 4 \\ \hline \end{array}$ are introduced. After a sufficient number of examples have been objectified concretely, "the pupils should write each of the 9 *zero facts* in multiplication."

During the process of learning the basic addition facts, Hickerson[6] recommended that "combinations containing zero not be included, since to the child, they have no meaning. 'Why,' the child asks, 'add nothing to something?' " This author agreed with Brueckner and Grossnickle in that the problem of the addition of zero first arises in column addition: $\begin{array}{r} 67 \\ +20 \\ \hline \end{array}$ "Since the only time 0 is added to a number is when the 0 is one of the digits of a number, there is no necessity for children to practice adding such expressions as $\begin{array}{r} 4 \\ +0 \\ \hline \end{array}$ and $\begin{array}{r} 0 \\ +8 \\ \hline \end{array}$."[7]

Morton[8] observed: "The *zero facts* are not used in life situations except when they occur in examples of numbers of two or more figures. In life, a pupil may have occasion to add 3 cents and 5 cents. . . . He will not have occasion to add 0 cents and 3 cents."

[1] Tobias Dantzig, *Number, The Language of Science* (New York: The Macmillan Company, 1939), p. 35.
[2] Charles Brumfiel, "Definitions in Arithmetic," THE ARITHMETIC TEACHER, 3: 192, November, 1956.
[3] Harry G. Wheat, *The Psychology and Teaching of Arithmetic* (Boston: D. C. Heath Company, 1937), p. 242.
[4] *Ibid.*, pp. 79–80.
[5] Leo J. Brueckner and Foster E. Grossnickle, *Making Arithmetic Meaningful* (Philadelphia: The John C. Winston Company, 1953), pp. 276–7.
[6] J. Allen Hickerson, *Guiding Children's Arithmetic Experiences* (New York: Prentice-Hall, Inc., 1952), p. 81.
[7] *Ibid.*, p. 93.
[8] Robert L. Morton, *Teaching Children Arithmetic* (New York: Silver Burdett Company, 1953), p. 145.

This writer implied that the *zero facts* are not used in subtraction in isolation either, for if a pupil "had 8 cents and spent the entire 8 cents, he would know that his money was all gone without performing the arithmetical operation of subtraction."

On the other hand, Hollister and Gunderson,[9] while basing their argument against teaching the *zero facts* in addition upon the difficulty of objectifying these facts, pointed out that it is "meaningful to use zero in subtraction problems in the first two grades." These writers asserted that first and second grade students can understand and would have occasion to use such facts as $3-3=0$.

According to Spitzer,[10] "Some writers and research workers have even concluded that the *zero facts* (such as $0+2$, $8+0$, and 8×0) are the most difficult to teach." With specific reference to multiplication: "It should be obvious that the teaching of the *zero facts* and the *one facts* should be delayed until the pupil has experienced a need for them in the multiplication of two-figured numbers."[11]

Wilson[12] summarized the research and observations of various writers from 1911 to 1927 concerned with the teaching of the *zero facts*. Chronologically, a changing attitude from a relative disregard to a specific need for instruction in the *zero facts* is discernible. According to Wilson, " . . . specific attention to zero in the teaching program rapidly eliminates the *zero errors*. . . . " Summarizing his opinions on this subject, this writer concluded:

> When attention is given to the *zero facts*, they are very quickly mastered. While it is no doubt better to teach all the zero combinations, they cause little or no trouble to pupils if the initial teaching of zero in combination is careful and thorough and at the right stage of mental level.[13]

The need for precise definitions of the types: $a+0=a$ and $a\times0=0$ [where a is a counting number] is undeniable, but too often this need is handled casually in the elementary school. The real question, however, appears to hinge upon the proper time to introduce these definitions. Should they be introduced in addition with the facts with sums less than ten, with the sums greater than ten, or when the addition of two-digit numbers is encountered? When should multiplication facts involving zero be taught?

It is a rather simple matter to teach a child to find the sum of $4+0$ or the product of 4×0 if we formulate the definitions: *A number plus zero is the number* and *A number times zero is zero*. However, is such manipulation really worthwhile before the need for this procedure occurs?

Accepting the proposition that arithmetic should be taught with meaning and understanding as the primary objectives, the value of proceeding from the concrete to the abstract is apparent. By using concrete objects the teacher can help the child gain a real and lasting understanding of the significance of addition and multiplication. Passing from the concrete, to the picture, to the semi-concrete stage in order until the abstract symbolism is presented, the teacher helps the child build a functional concept of arithmetic. But what happens to the concrete to abstract developmental sequence when the zero facts are included? Objectifying a null or empty group or set is obviously impossible. Perhaps, in exasperation, the teacher defines zero as 'nothing.' This in turn leads to the questionable technique of saying 'three plus nothing is three.' It appears to the child that zero *is* nothing in this and in other problems, but, ungrammatical as it may sound, zero is not 'nothing' in the true sense. Since it is not 'nothing,' it follows that it must be something, and the abstractness of the function and meaning of zero in our number system becomes apparent.

The consequence of thinking of zero as 'nothing' is epitomized by the elementary school teacher in one of the writer's methods

[9] George E. Hollister and Agnes G. Gunderson, *Teaching Arithmetic in Grades I and II* (New York: D. C. Heath and Company, 1954), pp. 12–15.

[10] Herbert F. Spitzer, *The Teaching of Arithmetic* (Boston: Houghton Mifflin Company, 1954), p. 55.

[11] *Ibid.*, p. 149.

[12] Guy M. Wilson, *Teaching the New Arithmetic* (New York: McGraw-Hill Book Company, Inc., 1951), pp. 130–1.

[13] *Ibid.*, p. 132.

classes who insisted that $\frac{1}{0}$ implies one over 'nothing.' Since "zero is nothing," this teacher casually erased the zero from the board, 'proving' to her satisfaction that the answer is *one!* Even recourse to the definition of division as the inverse of multiplication and a demonstration of the limit of $\frac{1}{a}$ as a approaches zero failed to convince this teacher that $\frac{1}{0}$ is anything but *one!*

In reality, a rather fine line of demarcation separates the real meaning of zero from 'nothing' as erroneously used in the preceding paragraph. According to Stokes:[14]

> Since numbers are symbols that represent the measure of magnitudes, if there is nothing to be measured there will be no number, just not any. Or when the magnitude is measurable in different-sized units in the same system and the use of a larger unit gives the value with nothing remaining to be measured by a smaller unit, it is appropriate to indicate this by the symbol 0, not any.
>
> There would be no question as to the meaning of 6 ft. 0 in. Likewise, 70 represents that something has been measured through the use of the ten's unit and it is measured without the need of the ones' unit.

Thus it appears that zero implies *not any of something* previously defined. The contention is submitted that 'nothing' and *not any of something* are not synonymous.

The significance of zero as a place holder and as a vital part of our number system appears, as many writers have stated, when we begin addition and multiplication of two-digit numbers. Until this time, there seems to be no real reason for introducing the zero facts in any of the basic operations. Perhaps one exception should be noted in the type of problem proposed by Hollister and Gunderson wherein three birds are in a nest, three fly away, how many are left?[15] The appearance of zero in the answer to this problem seems to place it in a different category from $0+a$, $a-0$, $a\times 0$, and $\frac{0}{a}$, where a is one of the counting numbers.

As Brumfiel implies, it is imperative that children be "taught arithmetic in such a manner that they have reasonable opportunity to grasp the definitions of the fundamental operations."[16] Definitions presented on a mature level to immature learners, however, might well perpetuate the rote learnings that have been shown to lack a firm and lasting psychological foundation.

In summary, there is a need to recognize the place of definitions in the teaching of elementary arithmetic, but the definitions should be introduced on a level appropriate to the maturity of the learner and in real situations where they are actually needed. Since it is difficult to present a significant problem at the elementary level in which zero is used in isolation, definitions involving the addition, subtraction, multiplication, and division of facts involving zero could well be delayed until two-digit numbers are involved.

Once the need for these definitions has arisen, the student should be given specific practice with these processes both in isolation and as integral parts of real problems. Generalizations from numerous contacts with zero should be derived from the student's observations, and understanding should evolve from the student's personal experience with these facts as they occur in real problems.

EDITOR'S NOTE. To a young child an "understanding-definition" develops through use in situations that have meaning to him. Hence his understanding of zero should be a development that includes the different uses and roles of zero. His "definition" should be one that suffices at the stage at which he is working but it should be one that may be enlarged and refined and of such a nature that it need not be denied or dis-allowed at a later stage. Frequently we may do an injustice to a child by trying to be too adult in our definitions. Likewise, we should not permit erroneous statements. This implies a fine line of judgment on the part of the teacher. What should the teacher say when a pupil in grade three casually asks about dividing some number by zero? It is a natural thing for an intelligent and inquisitive pupil to ask even though there is no "real-experience" situation to be represented by the process. Is it sufficient to say, "This we cannot do with our numbers"?

[14] C. Newton Stokes, *Teaching the Meanings of Arithmetic* (New York: Appleton-Century-Crofts, Inc., 1951), p. 410.

[15] Hollister and Gunderson, *op. cit.*, pp. 12–15.

[16] Brumfiel, *op. cit.*, p. 195.

Art and Arithmetic

ELIZABETH RAGLAND
Lexington Junior High School, Lexington, Ky.

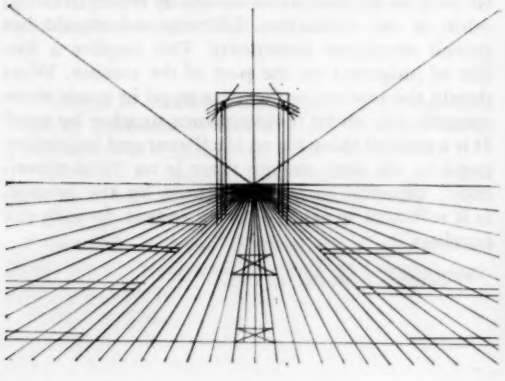
IT ALL started with the theme for the ninth grade class day progam. The subject was *Our Flower Garden.* Six homerooms were to represent six different flowers. The speeches and songs were to correspond with the central theme. It was suggested that the back auditorium wall would be a wonderful place for a scenic view.

The art teacher, Miss Juanita Todd, and one of the mathematics teachers, Miss Eizabeth Ragland, decided to try a correlation project. The scene would be such that the math students could apply some of their geometry and make guide lines for the art students who would do the other drawing and painting.

The math students took over first. The center of the wall was found. From the center point and from measured points along the floor and wall horizontal, vertical, and oblique lines were drawn. Arcs and angles were next. Measuring devices used in the classroom were of little use on a wall 14 feet by 24 feet. Ingenuity became a necessity. The wonders of string and chalk became apparent. Accurateness of measurement was a necessity for errors loomed large. Junior High geometry and arithmetic were put into practice.

The guide lines were drawn. The art students stepped in. Then more questions arose. Can certain flowers be this color? How many blossoms appear on one stem? Research was a must. Botany was studied with meaning. While art students painted, others did research.

Discussions arose. For example, nowhere in books could it be found whether flowers which had been painted a particular color on the wall could be that color. Early the next morning the one who had done the painting proved his point by bringing the *real flowers.*

While the arithmetic students showed perspective with lines, the art students showed perspective with shading. A feeling of depth and space showed clearly in the finished product.

Team work was wonderful. The fun of correlating art and arithmetic has a new meaning. Art students now see much arithmetic and arithmetic students now see more art.